Dynamic analysis of the damped trend proportional order-up-to policy

Qinyun Li[1] Gerard Gaalman[2] Stephen M. Disney[3]

[1]Logistics System Dynamics Group, Cardiff Business School, Cardiff University, UK

[2]Department of Operations, Faculty of Economics and Business, University of Groningen, THE NETHERLANDS

[3]Centre for Simulation, Analytics, and Modelling, University of Exeter Business School, UK

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Overview of our presentation

- History of proportional feedback control in inventory management
- Our research: Damped trend forecasting of ARIMA(1,1,2) demand and the proportional order-up-to (POUT) policy
- Is the bullwhip performance of the POUT policy always better than the OUT policy?

- **History of proportional feedback control in inventory management**
- Our research: Damped trend forecasting of ARIMA(1,1,2) demand and the proportional order-up-to policy
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History of proportional control in inventory management I

- Deziel and Eilon (1967) is one of the earliest studies of the proportional order-up-to (POUT) policy with discrete control theory.
- The continuous time APIOBPCS (John et al., 1994) model was developed based on the IOBPCS model (Towill, 1982).
- For more recent and more through literature reviews, Lin et al. (2017) and Ivanov et al. (2018).
- The POUT policy is a discrete time APIOBPCS model when the inventory feedback controller and the WIP feedback controller are taking the same value.
- POUT is able to alter the trade-off between inventory and capacity costs, and is effective in reducing the bullwhip effect.

History of proportional control in inventory management II

A steady stream of research on the POUT policy assumes stationary demand. Less attention has been given to non-stationary demand.

- Hosoda and Disney (2006) study the POUT policy's reaction to AR(1) demand in a three echelon supply chain.
- Gaalman (2006) studies the full-state OUT policy with ARMA(p,q) demand. The full-state policy has a proportional controller in both the feedback and the feed-forward paths in the OUT policy.
- Gaalman and Disney (2009) investigate both the POUT and the full-state OUT policy under ARMA(2,2) demand
- The POUT policy is used in i.i.d. demand, closed-loop setting by Zhou et al. (2017) and Cannella et al. (2021).
- Boute et al. (2022) considered non-stationary demand in dual sourcing setting with local SpeedFactories.

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The ARIMA(1,1,2) demand process

Assume a firm faces an ARIMA(1,1,2) demand process,

$$d_t = d_{t-1} + \phi(d_{t-1} - d_{t-2}) - \theta_1\eta_{t-1} - \theta_2\eta_{t-2} + \eta_t. \quad (1)$$

Here,

- ϕ is the auto-regressive parameter,
- θ_i is the moving average parameter at lag i,
- η_t is a sequence of i.i.d. random variables, with zero mean and finite variance $\mathbb{V}[\eta]$.

Variance of the ARIMA(1,1,2) demand, $\mathbb{V}[d_t]$

Due to the pole at unity, the variance of an ARIMA(1,1,2) demand process is infinite, Li et al. (2023).

The damped trend forecasting mechanism

Gardner and McKenzie (1985) provide the following recurrence form of the DT forecasting method:

$$\hat{a}_t = \alpha d_t + (1-\alpha)\left(\hat{a}_{t-1} + \gamma \hat{b}_{t-1}\right), \quad (2)$$

$$\hat{b}_t = \beta\left(\hat{a}_t - \hat{a}_{t-1}\right) + (1-\beta)\gamma \hat{b}_{t-1}, \quad (3)$$

$$\hat{d}_{t+k,t} = \hat{a}_t + \varphi[k]\,\hat{b}_t. \quad (4)$$

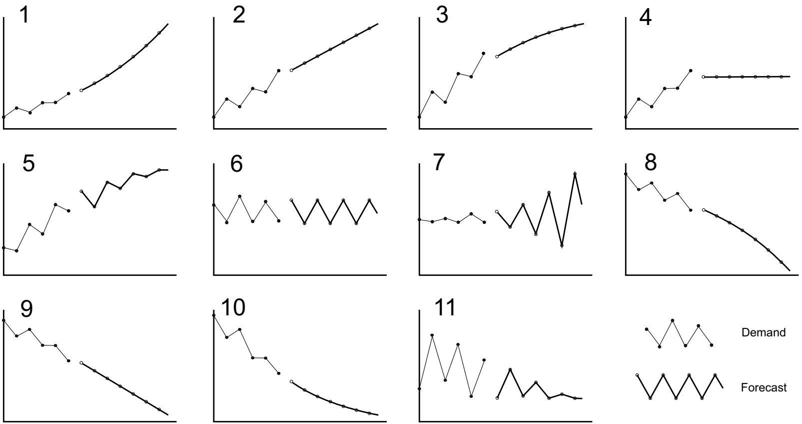
Here, $\hat{d}_{t+k,t}$ is the forecast of the demand k periods ahead, d_{t+k}, made at time t. $\hat{d}_{t+k,t}$ is the sum of a level, $\hat{a}_t$, and a trend, $\hat{b}_t$, component and

$$\varphi[k] = \sum_{i=1}^{k} \gamma^i = \frac{\gamma\left(1-\gamma^k\right)}{1-\gamma}. \quad (5)$$

$\{\alpha, \beta, \gamma\}$ are the DT forecasting parameters.

- α is a smoothing constant applied to the level $\hat{a}_t$.
- β is a smoothing constant applied to the trend $\hat{b}_t$.
- γ shapes the forecasts as they are projected into the future.

γ plays a significant role in DT forecasts, Disney (2024)



ϕ	Demand has an upward trend	Demand has a downward trend
$\phi > 1$	1) Convex, increasing	8) Concave, decreasing
$\phi = 1$	2) Linear, increasing	9) Linear, decreasing
$1 > \phi > 0$	3) Concave, increasing	10) Convex, decreasing
$\phi = 0$	4) Flat forecast, equal to current level	4) Flat forecast, equal to current level
$0 > \phi > -1$	5) Stable period two oscillations,upward trend	11) Stable period two oscillations, downward trend
$\phi = -1$	6) Critically stable period two oscillations	6) Critically stable period two oscillations
$-1 > \phi$	7) Unstable period two oscillations	7) Unstable period two oscillations

Equivalence of the damped trend and the ARIMA(1,1,2) demand process

Gardner and McKenzie (1985) also show the damped trend forecast produces a MMSE forecast of ARIMA(1,1,2) demand when

$$\left.\begin{aligned} \theta_1 &= 1 + \gamma - \alpha - \alpha\beta\gamma, \\ \theta_2 &= \gamma(\alpha - 1), \\ \phi &= \gamma. \end{aligned}\right\} \tag{6}$$

Given a set of ARIMA(1,1,2) parameters, perhaps identified from a real time series, we can solve the simultaneous equations in (6) for the damped trend parameters:

$$\left.\begin{aligned} \alpha &= \tfrac{\theta_2 + \phi}{\phi}, \\ \beta &= \tfrac{\phi^2 - \theta_2 - \theta_1\phi}{\theta_2\phi + \phi^2}, \\ \gamma &= \phi. \end{aligned}\right\} \tag{7}$$

The ARIMA(1,1,2) eigenvalues

The eigenvalues of the ARIMA(1,1,2) demand process can be used to make some bullwhip predictions. The eigenvalues can be identified from the z-transform transfer function of the ARIMA(1,1,2) demand process,

$$\frac{D_{ARIMA(1,1,2)}(z)}{\epsilon(z)} = \frac{z^2 - z\theta_1 - \theta_2}{z^2 - z(1+\phi) + \phi}. \tag{8}$$

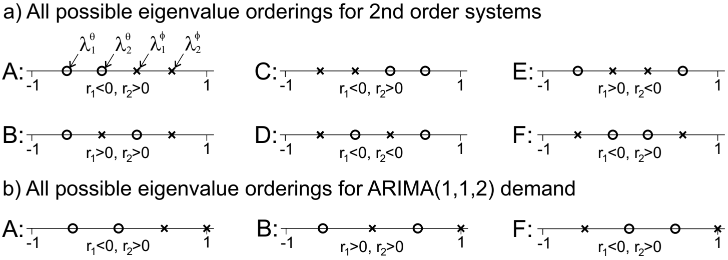
Eq. (8) has the following eigenvalues, Li et al. (2023):

$$\lambda_1^\theta = \frac{1}{2}\left(\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}\right), \quad \lambda_2^\theta = \frac{1}{2}\left(\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}\right),$$
$$\lambda_1^\phi = \phi, \quad \text{and} \quad \lambda_2^\phi = 1.$$

The moving average eigenvalues (λ_1^θ and λ_2^θ) are also known as the system zeros. The auto-regressive eigenvalues (λ_1^ϕ and λ_2^ϕ) are also known as the system poles.

Possible eigenvalue orderings

The eigenvalues (poles and zeros) of the dynamic system completely specify its linear dynamic response. The nature of the dynamic response is determined by the ordering (sequence) of the poles and zeros.



- Panel a: All possible eigenvalue ordering for second order systems, Gaalman et al. (2022).
- Panel b: Possible eigenvalue orderings for ARIMA(1,1,2) demand.

State space model of ARIMA(1,1,2) and its impulse response

We construct (1) into the state space model

$$d_{t+1} = \mathbf{D} d_t + \mathbf{G} \eta_t, \tag{9}$$

where

$$\mathbf{D} = \begin{pmatrix} 1+\phi & 1 \\ -\phi & 0 \end{pmatrix}, \ \mathbf{G} = \begin{pmatrix} 1+\phi-\theta_1 \\ -\phi-\theta_2 \end{pmatrix} = \begin{pmatrix} 1+\phi-(\lambda_1^\theta+\lambda_2^\theta) \\ -\phi+\lambda_1^\theta\lambda_2^\theta \end{pmatrix}. \tag{10}$$

We find the demand impulse response

$$p_{t+1} = r_1(\phi)^t + r_2, \ p_0 = 1, \tag{11}$$

and the impulse response for n-step ahead damped trend forecast

$$p_t(n) = r_1(\phi)^{t+n} + r_2, \tag{12}$$

where

$$r_1 = \frac{(\phi-\lambda_1^\theta)(\phi-\lambda_2^\theta)}{(\phi-1)}, \ r_2 = \frac{(1-\lambda_1^\theta)(1-\lambda_2^\theta)}{(1-\phi)}. \tag{13}$$

The OUT policy

The order decision is made at the end of period t, o_t, with a lead time k and a review period. That order is realised and influences the inventory at time $t+k+1$,

$$i_{t+k+1} = i_{t+k} + o_t - d_{t+k+1}. \tag{14}$$

Gaalman and Disney (2009) introduced the inventory gain component for the OUT policy,

$$E(0) = 1;\ E(k) = \sum_{j=0}^{k} p_j = 1 + \sum_{j=0}^{k-1} \mathbf{M}(\mathbf{D}^j)\mathbf{G}. \tag{15}$$

This helps us to write the inventory forecast

$$\hat{i}_{t+k+1,t+1} = \hat{i}_{t+k,t} + o_t - \hat{d}_{t+k+1,t} - E(k)\eta_{t+1}. \tag{16}$$

The proportional OUT policy (POUT)

The order generated by the POUT policy is

$$o_t = \begin{pmatrix} -f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i}_{t+k,t} \\ \hat{d}_{t+k+1,t} \end{pmatrix}. \tag{17}$$

f is the proportional feedback controller. $0 \leq f < 2$ is required for stability. When $f = 1$, the POUT policy (17) degenerates into the OUT policy. Then the forecast state space system for inventory and demand can be written as

$$\begin{pmatrix} \hat{i}_{t+k+1,t+1} \\ \hat{d}_{t+k+2,t+1} \end{pmatrix} = \begin{pmatrix} (1-f) & 0 \\ 0 & \mathbf{D} \end{pmatrix} \begin{pmatrix} \hat{i}_{t+k,t} \\ \hat{d}_{t+k+1,t} \end{pmatrix} + \begin{pmatrix} -E(k) \\ \mathbf{D}^k\mathbf{G} \end{pmatrix} \eta_{t+1}. \tag{18}$$

The OUT and POUT policy definition via difference equations

To avoid any doubt, the state space model for the POUT policy we have so far described is equivalent to

$$o_t = \hat{d}_{t+k+1,t} + f\left(i^\star - i_t + \sum_{j=1}^{k}(\hat{d}_{t+j,t} - o_{t-j})\right). \tag{19}$$

where:

- $\hat{d}_{t+k+1,t}$ is the forecasted demand in the period after the lead time
- $i^\star$ is a safety stock that can be set to achieve a target level of availability to minimise inventory holding and backlog costs via the newsvendor critical fractile.
- i is the current inventory level governed by the following balance equation

$$i_t = i_{t-1} + o_{t-k-1} - d_t \tag{20}$$

- $\sum_{j=1}^{k} \hat{d}_{t+j,t}$ is the target WIP, the sum of the forecasted demand over the lead time.
- $\sum_{j=1}^{k} o_{t-j}$ the actual WIP, the open orders that have been placed, but yet received.
- f is the proportional feedback controller, $0 \leq f < 2$.

Inventory variance

Matrix algebra will reveal the POUT policy's inventory variance is given by

$$\mathbb{V}[i_{t+k+1}] = \left(\frac{1}{f(2-f)}\right)(E(k))^2\mathbb{V}[\eta] + \sum_{j=0}^{k-1}(E(j))^2\mathbb{V}[\eta], \tag{21}$$

where $E(k)$ is the accumulated demand impulse p_t until $t = k$.

$$E(0) = 1;\ E(k) = \sum_{t=0}^{k} p_t \tag{22}$$

When $f = 1$, the POUT inventory variance expression degenerates into the OUT inventory variance expression.

$$\mathbb{V}[i_{t+k+1}] = \sum_{j=0}^{k}(E(j))^2\mathbb{V}[\eta]. \tag{23}$$

Remark. For both the OUT and POUT policies, the inventory variance is finite.

Demand and order variances

Demand variance in period $t + k + 1$ is

$$\mathbb{V}[d_{t+k+1}] = \mathbb{V}[\hat{d}_{t+k+1,t}] + \sum_{j=0}^{k} p_j^2 \mathbb{V}[\eta]. \tag{24}$$

Order variance in the POUT policy becomes

$$\begin{aligned}\mathbb{V}[o_t] =& \mathbb{V}[\hat{d}_{t+k+1,t}] + 2\sum_{j=1}^{2}\left(\frac{f}{1-(1-f)\lambda_l^{\phi}}\right) r_j(\lambda_j^{\phi})^k E(k)\mathbb{V}[\eta] \\ &+ \left(\frac{f}{2-f}\right)(E(k))^2\mathbb{V}[\eta].\end{aligned} \tag{25}$$

For $f = 1$, we have the order variance in the OUT policy,

$$\mathbb{V}[o_t] = \mathbb{V}[\hat{d}_{t+k+1,t}] + 2\sum_{j=1}^{2} r_j(\lambda_j^{\phi})^k E(k)\mathbb{V}[\eta]. \tag{26}$$

Remark. As $\mathbb{V}[\hat{d}_{t+k+1,t}] = \infty$, both $\mathbb{V}[d_{t+k+1}]$ and $\mathbb{V}[o_t]$ are infinite.

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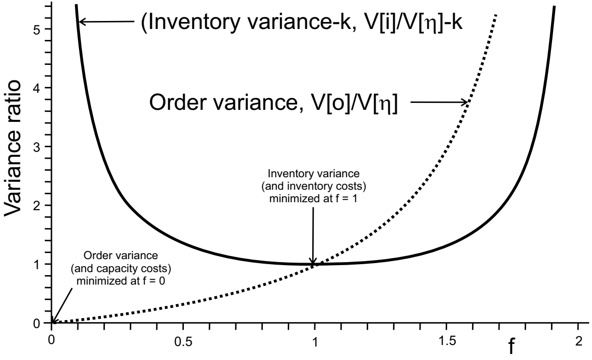
Under i.i.d. demand

- $\mathbb{V}[o] = \mathbb{V}[\eta]\frac{f}{2-f}$ is increasing convex in f with a minimum of zero at $f = 0$ and an asymptote to infinity as $f \to 2$. When $f = 1$, $\mathbb{V}[o] = \mathbb{V}[\eta]$
- $\mathbb{V}[i] = \mathbb{V}[\eta]\left(1 + k + \frac{(1-f)^2}{f(2-f)}\right)$ is always greater than $\mathbb{V}[\eta]$, always increases in the lead time k, convex in f with a minimum of $1 + k$ at $f = 1$, and has an asymptote to infinity when $f \leftarrow 0$ and $f \to 2$.



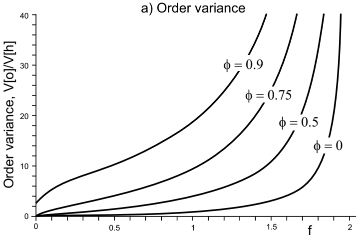
Impact of the proportional feedback controller under AR(1) demand

Hosoda and Disney (2006) show that the variance of the orders and inventory maintained by the POUT policy under AR(1) demand is given by

$$\frac{\mathbb{V}[o]}{\mathbb{V}[\eta]} = \frac{2f(\phi+1)(f+\phi-1)\phi^{k+1} - 2(f+\phi-1)^2\phi^{2k+2} - f(\phi+1)((f-1)\phi+1)}{(f-2)(\phi-1)^2(\phi+1)((f-1)\phi+1)} \tag{27}$$

and

$$\frac{\mathbb{V}[i]}{\mathbb{V}[\eta]} = \frac{(f-1)^2\left(\phi^{k+1}-1\right)^2}{(2-f)f(\phi-1)^2} + \frac{\phi\left(1-\phi^{k+1}\right)\left(\phi^{k+2}-\phi-2\right)+(k+1)\left(1-\phi^2\right)}{(1-\phi)^2\left(1-\phi^2\right)}. \tag{28}$$



Note: In the graph, $k = 3$.

The critical bullwhip measure, CB(k)

As the demand and order variance for ARIMA(1,1,2) is infinite, we measure bullwhip as the difference between the order variance and the demand variance,

$$CB(k) = \mathbb{V}[o_t] - \mathbb{V}[d_{t+k+1}]. \tag{29}$$

To compare the bullwhip between DT-OUT and DT-POUT policies, we can measure $CB(k)^O - CB(k)^P = \mathbb{V}[o_t]^O - \mathbb{V}[o_t]^P$,

$$CB(k)^O - CB(k)^P = 2(1-f)E(k)\left(\frac{E(k)}{2-f} + \frac{r_1(1-\phi)(\phi)^k}{1-(1-f)\phi}\right)\mathbb{V}[\eta]. \tag{30}$$

Remark. When $\phi < 0$, an odd-even lead time effect can be seen in (30).

When the bullwhip effect is increasing in the lead time

Theorem (Necessary-sufficient condition for an increasing bullwhip effect in the lead time, Gaalman et al. (2022))

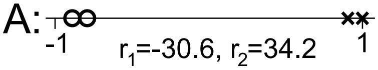
$CB[k]$ is always positive and increasing in the lead time $\forall k$ iff $\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_{k+1}\} > 0$.

Proof. *$CB[k]$ is positive and increasing in k if $CB[0] > 0$ and $\forall k$, $CB[k] - CB[k-1] > 0$. Note always, $\tilde{d}_0 = 1$. $CB[0] = \left(\sum_{j=0}^{1} \tilde{d}_j\right)^2 - \sum_{t=0}^{1} \tilde{d}_t^2 = 2\tilde{d}_0\tilde{d}_1$ is positive if additionally $\tilde{d}_1 > 0$. $CB[1] - CB[0] = 2(\tilde{d}_0 + \tilde{d}_1)\tilde{d}_2$ is positive if additionally $\tilde{d}_2 > 0$. $CB[2] - CB[1] = 2(\tilde{d}_0 + \tilde{d}_1 + \tilde{d}_2)\tilde{d}_3$ is positive if additionally $\tilde{d}_3 > 0$. This process can be continued $\forall k$, indicating that bullwhip is always present and increasing in the lead-time iff the demand impulse response is positive for all t.* □

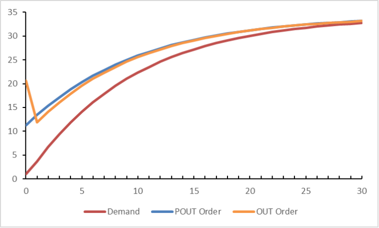
This theorem shows that bullwhip is always present and always increasing in the lead-time if, and only if, the demand impulse response is positive for all t; that is, $CB[k]$ is increasing in k iff $\forall t$, $\tilde{d}_t > 0$.

For Type A ARIMA(1,1,2) demand

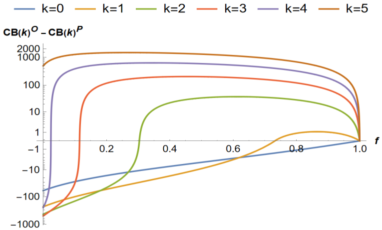
$$\phi = 0.9,\ \ \theta_1 = -1.7,\ \ \theta_2 = -0.72, \lambda_1^\phi = 0.9, \lambda_2^\phi = 1, \lambda_1^\theta = -0.9, \lambda_2^\theta = -0.8$$



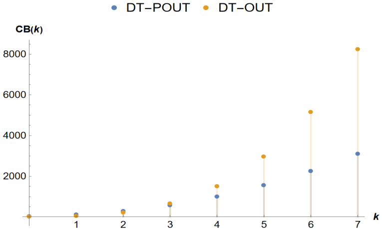
Eigenvalues



Impulse response $(k = 2,\ f = 0.16)$



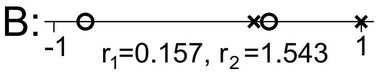
Bullwhip comparison



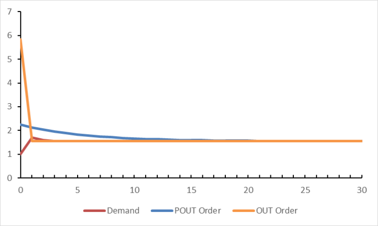
Impact of lead time $(f = 0.16)$

For Type B ARIMA(1,1,2) demand

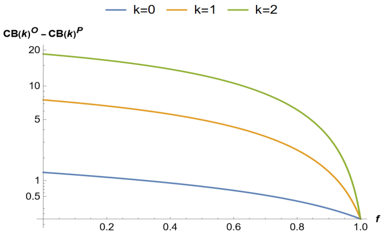
$$(\phi = 0.3,\ \theta_1 = -0.4,\ \theta_2 = 0.32, \lambda_1^{\phi} = 0.3,\ \lambda_2^{\phi} = 1,\ \lambda_1^{\theta} = -0.8,\ \lambda_2^{\theta} = 0.4,)$$



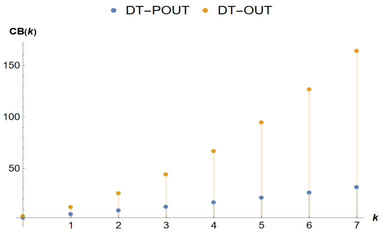
Eigenvalues



Impulse response $(k = 2,\ f = 0.16)$



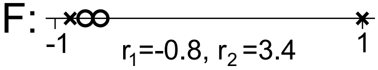
Bullwhip comparison



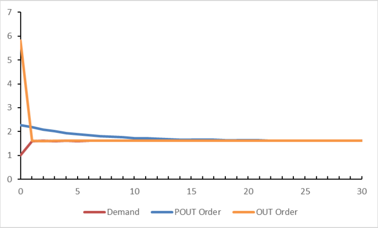
Impact of lead time $(f = 0.16)$

For Type F ARIMA(1,1,2) demand

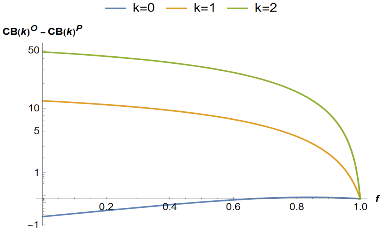
$$(\phi = 0.1,\ \theta_1 = -1.5,\ \theta_2 = -0.56, \lambda_1^{\phi} = -0.9,\ \lambda_2^{\phi} = 1,\ \lambda_1^{\theta} = -0.8,\ \lambda_2^{\theta} = -0.7,)$$



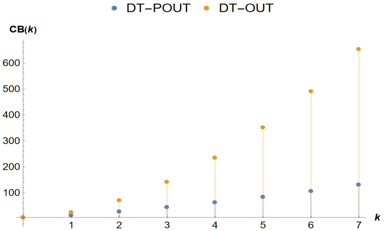
Eigenvalues



Impulse response $(k = 2,\ f = 0.16)$



Bullwhip comparison



Impact of lead time $(f = 0.16)$

Concluding remarks

- We have studied the bullwhip effect in both OUT and POUT policies with optimal forecasts for ARIMA(1,1,2) demand.
- We quantified the bullwhip effect for all possible ARIMA(1,1,2) demand under the OUT and POUT policy.
- Our analysis shows that conventional values for proportional controller can sometimes (for Type A and Type F ARIMA(1,1,2) demand) create a larger bullwhip than the DT-OUT policy.
- For Type B, all values of $0 < f < 1$ can reduce the bullwhip effect in the DT-POUT policy.
- Based on the eigenvalues of the demand process, we provide the conditions where the DT-OUT policy has better bullwhip performance than the DT-POUT policy.

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