DISCRETE CONTROL THEORY

Stephen Disney

27 August 2019

Overview

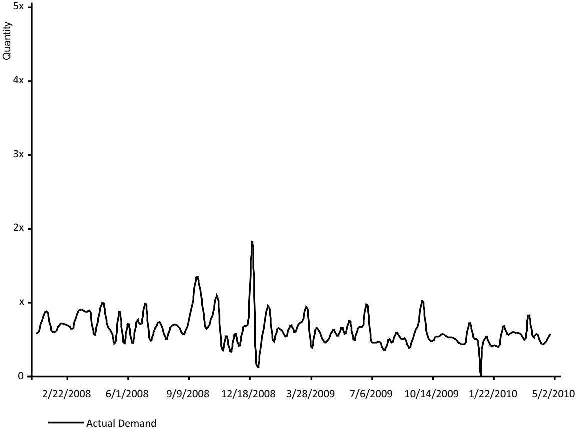
- I **believe** supply chains can be dynamically designed to reduce cost, improve product availability and create a stable working environment for employees and suppliers.

Overview

- I **believe** supply chains can be dynamically designed to reduce cost, improve product availability and create a stable working environment for employees and suppliers.
- Discrete control theory, z-transforms in particular, can provide a toolkit for understanding the dynamics of supply chain replenishment decisions.
- We can understand stochastic performance via Tyspkin's Relation.
- We can design supply chain replenishment policies to improve their economic performance via newsvendor techniques.

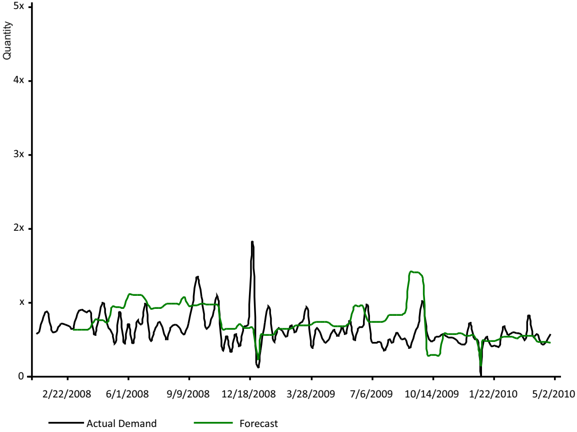
Supply chain demand can be variable

Here is a real life demand pattern.



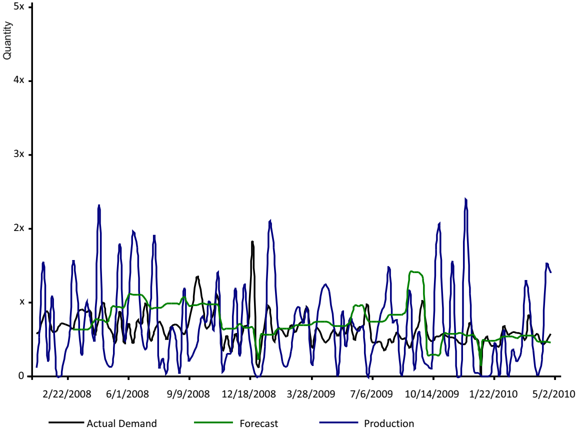
Company forecasts can be biased

This is how the company forecasted the demand pattern one week ahead.



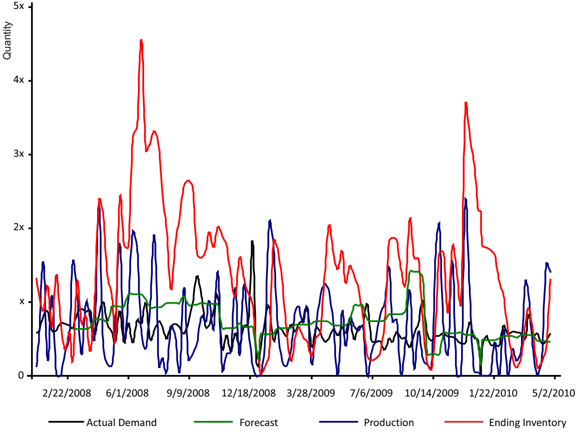
Forecasting and replenishment rules create bullwhip

The production was 7 times more variable than demand.



Creating wide fluctuations in inventory levels

The inventory was 16 times more variable then demand.



The order-up-to policy

- The order-up-to (OUT) policy is a popular policy for placing production and replenishment orders to maintain control over an inventory.
- The OUT policy is available native in many commercial ERP/MRP systems; often used to to schedule high volume, long life products.
- The OUT policy is defined by an inventory balance equation

$$i_t = i_{t-1} + q_{t-L-1} - d_t \tag{1}$$

and an ordering rule,

$$q_t = \hat{d}_{t,t+L+1} + (i^\star - i_t) + \sum_{j=1}^{L} (\hat{d}_{t,t+j} - q_{t-i}) \tag{2}$$

- **Notation.** i_t is the inventory at time t. q_{t-L-1} is the order placed the physical lead time L and review period $(+1)$ ago. d_t is the demand. $\hat{d}_{t,t+x}$ is a forecast of demand, made at time t of demand in period $t+x$.

Yakov Tsypkin: Discrete Laplace transform

Yakov Zalmanovitch Tsypkin

"He is considered to be the father of pulsed systems in the East. In a series of papers in 1949 and 1950, he extensively developed the discrete Laplace transform (z-transform and modified z-transform) which he applied to the study of pulsed systems. This work culminated in his classic book in this field in 1958." (Bissell, 1992).

September 19, 1919 - December 2, 1997

Understanding dynamic behaviour with z-transforms

- z-transforms are the discrete-time analogue of the Laplace transform.
- Linear discrete-time systems are governed by linear difference equations.
- System analysis requires the solution of these systems for a given input function; in the time, or the frequency, domain.
- Time domain solutions require convolution, but frequency domain solutions obtained with the z-transform require only addition and multiplication.
- The z-transform is defined by,

$$X[z] = Z[x_t] = \sum_{t=0}^{\infty} x_t z^{-t}. \tag{3}$$

Understanding the z-transform: Impulse response

Linearity implies that the system output y_t for any input function x_t is fully described by the impulse response function g_t, which is the solution of the system's difference equation when the input is the unit impulse function $\delta[t]$; $\delta[t=0]=1$ and $\delta[t\neq 0]=0$.

Time, t	$x_t=\delta[t]$	$y_t=g_t$
0	1	1
1	0	0
2	0	0
3	0	0
$\vdots$	0	0

$$\begin{aligned} G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\ &= (1\times z^{0})+(0\times z^{-1})+(0\times z^{-2})+(0\times z^{-3})+\dots \\ &= 1 \end{aligned}$$

Understanding the z-transform: Delay operator

The output of the system can be time shifted one period into the future with the delay operator, z^{-1}.

Time, t	$x_t = \delta[t]$	$y_t = g_t$
0	1	0
1	0	1
2	0	0
3	0	0
$\vdots$	0	0

$$\begin{aligned} G[z] =& \sum_{t=0}^{\infty} g_t z^{-t} \\ =&(0 \times z^0) + (1 \times z^{-1}) + (0 \times z^{-2}) + (0 \times z^{-3}) + \ldots \\ =&z^{-1} \end{aligned}$$

Understanding the z-transform: Scaled responses

We can scale an output with simple multiplication. Here we have combined a scaling operation with a delay.

Time, t	$x_t = \delta[t]$	$y_t = g_t$
0	1	0
1	0	0.5
2	0	0
3	0	0
$\vdots$	0	0

$$
\begin{aligned}
G[z] =& \sum_{t=0}^{\infty} g_t z^{-t} \\
=& (0 \times z^0) + (0.5 \times z^{-1}) + (0 \times z^{-2}) + (0 \times z^{-3}) + \ldots \\
=& 0.5z^{-1}
\end{aligned}
$$

Understanding the z-transform: Time integration

We can integrate the time domain response with the integration operator $\frac{z}{z-1}$. This is really useful for determining the inventory response.

Time, t	$x_t = \delta[t]$	$y_t = g_t$
0	1	1
1	0	1
2	0	1
3	0	1
$\vdots$	0	1

$$\begin{aligned} G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\ &= (1 \times z^0) + (1 \times z^{-1}) + (1 \times z^{-2}) + (1 \times z^{-3}) + \ldots \\ &= \sum_{i=0}^{\infty} z^{-i} = \frac{z}{z-1} \end{aligned}$$

Linear systems and the impulse response

- Linearity implies that the system output y_t for any input function x_t is fully described by the impulse response function g_t.
- Any input function is the superposition of scaled and delayed impulse functions, its output is similarly the superposition of scaled and delayed impulse response functions.
- Equivalently, the output is the convolution of the input and impulse response:

$$y_t = \sum_{i=0}^{t} x_i g_{t-i} \longleftrightarrow Y[z] = X[z]G[z]. \tag{4}$$

- The Z-transform of the impulse response function $G(z)$ is called the *transfer function* where convolutions are transformed into simple multiplications.

The OUT policy under i.i.d. demand and minimum mean squared error (MMSE) forecasting

- Consider an i.i.d. random demand pattern (uncorrelated, drawn from a normal distribution with mean μ and variance σ^2) exists.
- MMSE forecasts, for all periods ahead, of an i.i.d. demand pattern are simply the mean demand μ,
- The OUT policy difference equations become

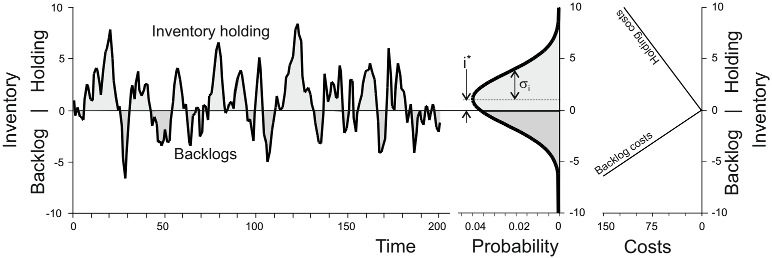
$$q_t = \mu + (i^\star - i_t) + (\mu L - \sum_{j=1}^{L} q_{t-i}) \tag{5}$$

- The order-up-to policy is known to be the optimal policy for minimizing inventory holding and backlog costs.

Economics of the OUT policy: Inventory costs

Per period, per unit, inventory holding (h) and backlog costs (b) exist:

$$C_t^i = h[i_t]^+ + b[-i_t]^+. \tag{6}$$



Using standard newsvendor techniques (Churchman et al., 1957) a safety stock of

$$i^\star = \sigma_i z_i;\;\; z_i = \Phi^{-1}\left[b/(b+h)\right] \tag{7}$$

results in the minimal long-run average per period inventory cost

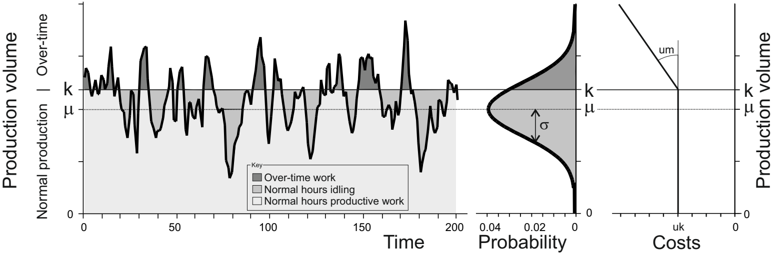
$$C_i^\star = \sigma_i (h+b)\phi\left[z_i\right]. \tag{8}$$

Note: Here $\{\phi[\cdot], \Phi^{-1}[\cdot]\}$ is the p.d.f., and inverse of the c.d.f., of the standard normal distribution respectively.

Economics of the OUT policy: Capacity costs

A per period production cost C_t^q, with a nominal hours unit cost of u and flexible overtime at unit cost um, where m is the overtime multiplier:

$$C_t^q = uk + um[q_t - k]^+. \tag{9}$$



Hosoda and Disney (2012) show the optimal capacity $k^\star$ is given by

$$k^\star = \mu + \sigma_q z_q; \quad z_q = \Phi^{-1}\left[(m-1)/m\right]. \tag{10}$$

The associated minimal long-run average production costs is

$$C_q^\star = u\mu + um\sigma_q \phi[z_q]. \tag{11}$$

To minimize the sum of inventory and capacity costs we propose the proportional order-up-to policy

- The proportional OUT policy, introduces a feedback controller $(1-\alpha)$ into the inventory and WIP feedback loops.
- The POUT policy difference equations become

$$q_t = \mu + (1-\alpha)(i^\star - i_t + \mu L - \sum_{j=1}^{L} q_{t-i}) \tag{12}$$

- $0 < \alpha \leq 1$ allows the smoothing of production orders, Disney and Towill (2003), but at the expense of increased inventory costs.
- The inventory balance equation remains the same

$$i_t = i_{t-1} + q_{t-L-1} - d_t \tag{13}$$

- This heuristic is easily implemented in planning spreadsheets, commercial ERP systems (including SAP), and bespoke IT systems.
- Performance of the POUT policy has been shown to be within 1% of the optimal *dual base stock* policy, Boute et al. (2019).

The variance ratio and the sum of the squared impulse response

- Both the standard deviation of the inventory and the standard deviation of the orders are required to evaluate the costs.
- The impulse response function directly allows the exact computation of the variance of the system output:

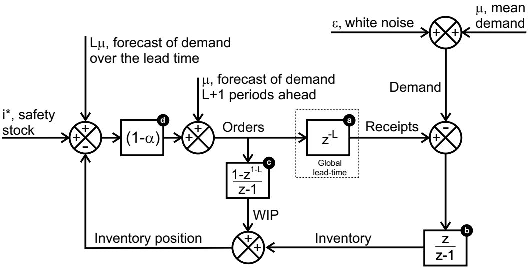
Lemma (Typskin's Relation)

If the input x_t to a linear system with impulse response function g_t is an i.i.d. random process with variance σ_x^2, then the long-run variance of the output y_t is

$$\sigma_y^2 = \sigma_x^2 \sum_{t=0}^{\infty} (g_t)^2, \tag{14}$$

(Tsypkin, 1964).

Block diagram and transfer functions of the proportional order-up-to policy



- Rearranging the block diagram for transfer function of the orders,

$$Q[z] = \frac{bd}{1+abd+cd} = \frac{z-\alpha z}{z-\alpha}. \tag{15}$$

- Similarly, the inventory transfer function is

$$I[z] = (Q[z]a-1)b = \frac{z\left(\alpha - z^{1-L}\left(\alpha + z^{L} - 1\right)\right)}{(z-1)(z-\alpha)} \tag{16}$$

Obtaining the variance of the production orders

- Taking the inverse z-transform of the order transfer function gives the order impulse response function

$$q_t = Z^{-1}\left[\frac{z-\alpha z}{z-\alpha}\right] = (1-\alpha)\alpha^t. \tag{17}$$

- Using Tyspkin's Relation the order variance is obtained

$$\sigma_q^2 = \sigma^2 \sum_{t=0}^{\infty}((1-\alpha)\alpha^t)^2 = \sigma^2\left(\frac{1-\alpha}{1+\alpha}\right). \tag{18}$$

Obtaining the variance of the inventory

- Taking the inverse z-transform of the inventory transfer function gives the order impulse response function

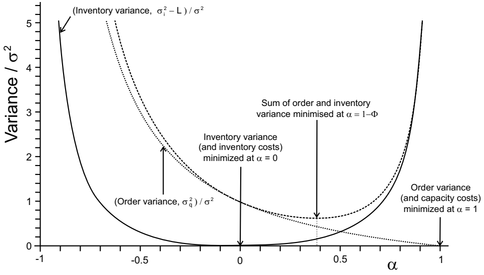
$$
\begin{aligned}
i_t &= Z^{-1}\left[\frac{z\left(\alpha - z^{1-L}\left(\alpha + z^{L} - 1\right)\right)}{(z-1)(z-\alpha)}\right] && (19)\\
&= \frac{\left(\alpha^{L} - \alpha^{L+t+1} + h[t-1]\alpha^{L+1}\left(\alpha^{t}-1\right) + h[t-L](\alpha-1)\left(\alpha^{L}-\alpha^{t+1}\right)\right)}{\alpha^{L}(\alpha-1)} && (20)
\end{aligned}
$$

  Here $h[t]$ is the Unit Step function. That is, $h[t<0]=0$, otherwise $h[t\geq 0]=1$.

- Using Tyspkin's Relation the inventory variance is given by

$$\sigma_i^2 = \sigma^2\left(\sum_{t=0}^{L-1}(-1)^2 + \sum_{t=L}^{\infty}\left(-\alpha^{t-L+1}\right)^2\right) = \sigma^2\left(L + \frac{\alpha^2}{1-\alpha^2}\right) \qquad (21)$$

The golden feedback controller, α_Φ: Disney et al. (2004, 2007)



- Take the derivative of the sum of the variances

$$\frac{d[\sigma_q^2 + \sigma_i^2]}{d\alpha} = -\frac{2((3-\alpha)\alpha - 1)}{\left(\alpha^2 - 1\right)^2} \tag{22}$$

- Use the first order condition to find the golden α

$$\alpha_\Phi = \frac{3-\sqrt{5}}{2} = 2 - \frac{1+\sqrt{5}}{2} = 1 - \Phi \tag{23}$$

The total cost function

For the OUT policy with associated inventory and production variances σ_i^2 and σ_q^2, and a proportional feedback controller α, the total costs are given by

$$C = C_i^\star + C_q^\star = \sigma_i (h + b)\phi\left[z_i\right] + \sigma_q u m \phi\left[z_q\right] + u\mu. \tag{24}$$

This can be re-written as

$$C = \varphi(\sigma_i + \lambda(\sigma_q - \sigma_i)) + u\mu, \tag{25}$$

where $\lambda \in [0, 1]$ indicates the capacity intensity and $\varphi \geq 0$ the scaling factor,

$$\lambda = um\phi[z_q]/\varphi \text{ and } \varphi = (h + b)\phi[z_i] + um\phi[z_q]. \tag{26}$$

The cost optimal feedback controller

Proposition 1 (Optimal $\alpha^\star$). Patterned on Hedenstierna and Disney (2018)

For a given capacity intensity $\lambda \in [0,1]$, the optimal smoothing $\alpha^\star = \lambda$.

Proof of Proposition 1 (Optimal $\alpha^\star$) The derivative of (25) with respect to α is

$$\frac{dC}{d\alpha} = \varphi\left(\frac{(1-\lambda)\frac{d\sigma_i^2}{d\alpha}}{2\sigma_i} + \frac{\lambda\frac{d\sigma_q^2}{d\alpha}}{2\sigma_q}\right). \tag{27}$$

The standard derivations have been presented previously, the derivatives are

$$\frac{d\sigma_i^2}{d\alpha} = \frac{2\alpha}{\left(\alpha^2-1\right)^2}, \text{ and } \frac{d\sigma_q^2}{d\alpha} = -\frac{2}{(\alpha+1)^2}. \tag{28}$$

After substitution and simplification, the first order condition then reveals the stated relation, $\alpha^\star = \lambda$.

Concluding remarks

- Discrete control theory, z-transforms in particular, can provide a toolkit for understanding the dynamics of supply chain replenishment decisions.
- We can understand stochastic performance via Tyspkin's Relation.
- We can design supply chain replenishment policies to improve their economic performance via newsvendor techniques.

Concluding remarks

- Discrete control theory, z-transforms in particular, can provide a toolkit for understanding the dynamics of supply chain replenishment decisions.
- We can understand stochastic performance via Tyspkin's Relation.
- We can design supply chain replenishment policies to improve their economic performance via newsvendor techniques.
- Supply chains **can** be dynamically designed to reduce cost, improve product availability and create a stable working environment for employees and suppliers.

Thank you for listening

DISCRETE CONTROL THEORY

Stephen M. Disney
DisneySM@cardiff.ac.uk

Bibliography

Bissell, C. 1992. Yakov Tyspkin: A Russian life in control. *IEE Review* **Sept** 313–316.

Boute, R.N., S.M. Disney, J.A. Van Mieghem. 2019. Dual sourcing and smoothing under non-stationary demand time series: Re-shoring with speedfactories. *Under review* 32 pp.

Churchman, C. W., R. L. Ackoff, E. L. Arnoff. 1957. *Introduction to Operations Research*. John Wiley & Sons, New York.

Disney, S. M., D. R. Towill. 2003. On the bullwhip and inventory variance produced by an ordering policy. *Omega* **31**(3) 157–167.

Disney, S.M., D.R. Towill, W. van de Velde. 2004. Variance amplification and the golden ratio in production and inventory control. *International Journal of Production Economics* **90**(3) 295 – 309.

Disney, Stephen M., Ingrid Farasyn, Marc R. Lambrecht, Denis R. Towill, Wim Van De Velde. 2007. Controlling bullwhip and inventory variability with the golden smoothing rule. *European Journal of Industrial Engineering* **1**(3) 241–265.

Hedenstierna, C. P. T., S. M. Disney. 2018. Avoiding the capacity cost trap: Three means of smoothing under cyclical production planning. *International Journal of Production Economics* **201** 149–162.

Hosoda, T., S. M. Disney. 2012. On the replenishment policy when the market demand information is lagged. *International Journal of Production Economics* **135**(1) 458–467.

Tsypkin, Y. Z. 1964. *Sampling Systems Theory and its Application*, vol. 2. Pergamon Press, Oxford.