

# DISCRETE CONTROL THEORY

Stephen Disney

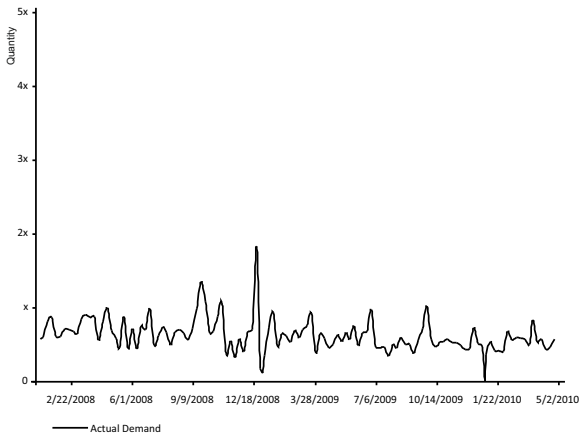
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- We can understand stochastic performance via Tyspkin's Relation.
- We can design supply chain replenishment policies to improve their economic performance via newsvendor techniques.

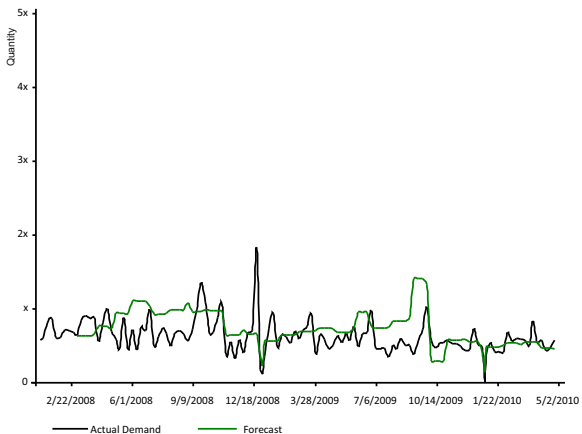
# Supply chain demand can be variable

Here is a real life demand pattern.



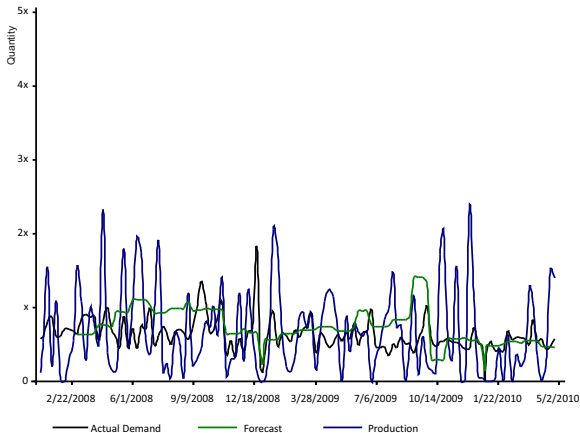
# Company forecasts can be biased

This is how the company forecasted the demand pattern one week ahead.



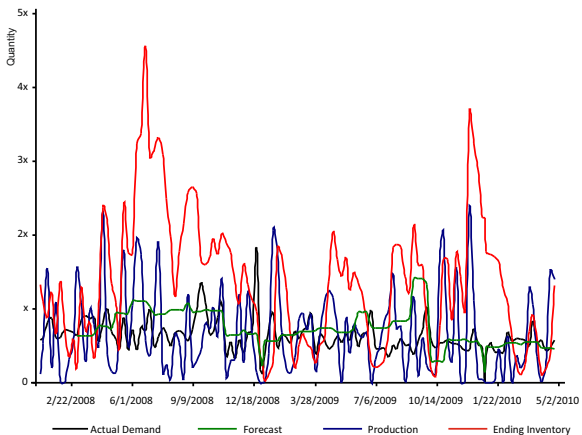
# Forecasting and replenishment rules create bullwhip

The production was 7 times more variable than demand.



# Creating wide fluctuations in inventory levels

The inventory was 16 times more variable than demand.



# The order-up-to policy

- The order-up-to (OUT) policy is a popular policy for placing production and replenishment orders to maintain control over an inventory.
- The OUT policy is available native in many commercial ERP/MRP systems; often used to schedule high volume, long life products.
- The OUT policy is defined by an inventory balance equation

$$i_t = i_{t-1} + q_{t-L-1} - d_t \quad (1)$$

and an ordering rule,

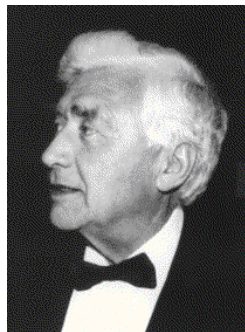
$$q_t = \hat{d}_{t,t+L+1} + (i^* - i_t) + \sum_{j=1}^L (\hat{d}_{t,t+j} - q_{t-j}) \quad (2)$$

- **Notation.**  $i_t$  is the inventory at time  $t$ .  $q_{t-L-1}$  is the order placed the physical lead time  $L$  and review period  $(+1)$  ago.  $d_t$  is the demand.  $\hat{d}_{t,t+x}$  is a forecast of demand, made at time  $t$  of demand in period  $t+x$ .

# Yakov Tsytkin: Discrete Laplace transform

## Yakov Zalmanovitch Tsytkin

“He is considered to be the father of pulsed systems in the East. In a series of papers in 1949 and 1950, he extensively developed the discrete Laplace transform (z-transform and modified z-transform) which he applied to the study of pulsed systems. This work culminated in his classic book in this field in 1958.”  
(Bissell, 1992).



September 19, 1919 - December 2, 1997

# Understanding dynamic behaviour with z-transforms

- z-transforms are the discrete-time analogue of the Laplace transform.
- Linear discrete-time systems are governed by linear difference equations.
- System analysis requires the solution of these systems for a given input function; in the time, or the frequency, domain.
- Time domain solutions require convolution, but frequency domain solutions obtained with the z-transform require only addition and multiplication.
- The z-transform is defined by,

$$X[z] = Z[x_t] = \sum_{t=0}^{\infty} x_t z^{-t}. \quad (3)$$

# Understanding the z-transform: Impulse response

Linearity implies that the system output  $y_t$  for any input function  $x_t$  is fully described by the impulse response function  $g_t$ , which is the solution of the system's difference equation when the input is the unit impulse function  $\delta[t]$ ;  $\delta[t = 0] = 1$  and  $\delta[t \neq 0] = 0$ .

Time, $t$	$x_t = \delta[t]$	$y_t = g_t$
0	1	1
1	0	0
2	0	0
3	0	0
$\vdots$	0	0

$$\begin{aligned} G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\ &= (1 \times z^0) + (0 \times z^{-1}) + (0 \times z^{-2}) + (0 \times z^{-3}) + \dots \\ &= 1 \end{aligned}$$

# Understanding the z-transform: Delay operator

The output of the system can be time shifted one period into the future with the delay operator,  $z^{-1}$ .

Time, $t$	$x_t = \delta[t]$	$y_t = g_t$
0	1	0
1	0	1
2	0	0
3	0	0
$\vdots$	0	0

$$\begin{aligned}G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\ &= (0 \times z^0) + (1 \times z^{-1}) + (0 \times z^{-2}) + (0 \times z^{-3}) + \dots \\ &= z^{-1}\end{aligned}$$

# Understanding the z-transform: Scaled responses

We can scale an output with simple multiplication. Here we have combined a scaling operation with a delay.

Time, $t$	$x_t = \delta[t]$	$y_t = g_t$
0	1	0
1	0	0.5
2	0	0
3	0	0
$\vdots$	0	0

$$\begin{aligned}G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\ &= (0 \times z^0) + (0.5 \times z^{-1}) + (0 \times z^{-2}) + (0 \times z^{-3}) + \dots \\ &= 0.5z^{-1}\end{aligned}$$

# Understanding the z-transform: Time integration

We can integrate the time domain response with the integration operator  $\frac{z}{z-1}$ . This is really useful for determining the inventory response.

Time, $t$	$x_t = \delta[t]$	$y_t = g_t$
0	1	1
1	0	1
2	0	1
3	0	1
$\vdots$	0	1

$$\begin{aligned}G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\&= (1 \times z^0) + (1 \times z^{-1}) + (1 \times z^{-2}) + (1 \times z^{-3}) + \dots \\&= \sum_{i=0}^{\infty} z^{-i} = \frac{z}{z-1}\end{aligned}$$

# Linear systems and the impulse response

- Linearity implies that the system output  $y_t$  for any input function  $x_t$  is fully described by the impulse response function  $g_t$ .
- Any input function is the superposition of scaled and delayed impulse functions, its output is similarly the superposition of scaled and delayed impulse response functions.
- Equivalently, the output is the convolution of the input and impulse response:

$$y_t = \sum_{i=0}^t x_i g_{t-i} \longleftrightarrow Y[z] = X[z]G[z]. \quad (4)$$

- The  $Z$ -transform of the impulse response function  $G(z)$  is called the *transfer function* where convolutions are transformed into simple multiplications.

# The OUT policy under i.i.d. demand and minimum mean squared error (MMSE) forecasting

- Consider an i.i.d. random demand pattern (uncorrelated, drawn from a normal distribution with mean  $\mu$  and variance  $\sigma^2$ ) exists.
- MMSE forecasts, for all periods ahead, of an i.i.d. demand pattern are simply the mean demand  $\mu$ ,
- The OUT policy difference equations become

$$q_t = \mu + (i^* - i_t) + (\mu L - \sum_{j=1}^L q_{t-j}) \quad (5)$$

- The inventory balance equation remains the same

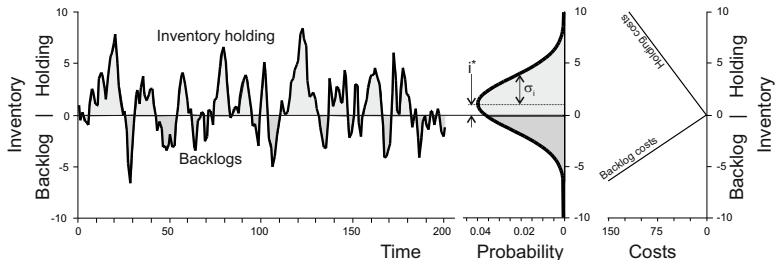
$$i_t = i_{t-1} + q_{t-L-1} - d_t \quad (6)$$

- The order-up-to policy is known to be the optimal policy for minimizing inventory holding and backlog costs.

# Economics of the OUT policy: Inventory costs

Per period, per unit, inventory holding ( $h$ ) and backlog costs ( $b$ ) exist:

$$C_t^i = h[i_t]^+ + b[-i_t]^+. \quad (7)$$



Using standard newsvendor techniques (Churchman et al., 1957) a safety stock of

$$i^* = \sigma_i z_i; \quad z_i = \Phi^{-1} [b/(b + h)] \quad (8)$$

results in the minimal long-run average per period inventory cost

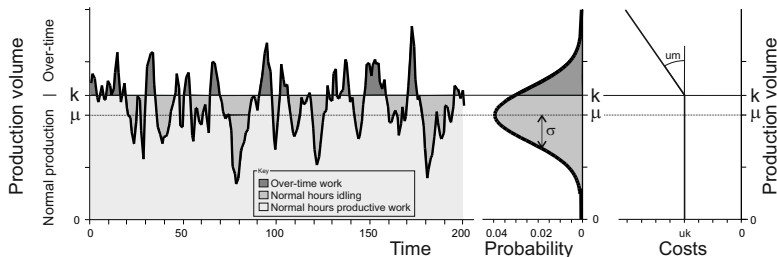
$$C_i^* = \sigma_i (h + b) \phi [z_i]. \quad (9)$$

Note: Here  $\{\phi[\cdot], \Phi^{-1}[\cdot]\}$  is the p.d.f., and inverse of the c.d.f., of the standard normal distribution respectively.

# Economics of the OUT policy: Capacity costs

A per period production cost  $C_t^q$ , with a nominal hours unit cost of  $u$  and flexible overtime at unit cost  $um$ , where  $m$  is the overtime multiplier:

$$C_t^q = uk + um[q_t - k]^+. \quad (10)$$



Hosoda and Disney (2012) show the optimal capacity  $k^*$  is given by

$$k^* = \mu + \sigma_q z_q; \quad z_q = \Phi^{-1} [(m-1)/m]. \quad (11)$$

The associated minimal long-run average production costs is

$$C_q^* = u\mu + um\sigma_q\phi[z_q]. \quad (12)$$

# To minimize the sum of inventory and capacity costs we propose the proportional order-up-to policy

- The proportional OUT policy, introduces a feedback controller  $(1 - \alpha)$  into the inventory and WIP feedback loops.
- The POUT policy difference equations become

$$q_t = \mu + (1 - \alpha)(i^* - i_t + \mu L - \sum_{j=1}^L q_{t-j}) \quad (13)$$

- $0 < \alpha \leq 1$  allows the smoothing of production orders, Disney and Towill (2003), but at the expense of increased inventory costs.
- This heuristic is easily implemented in planning spreadsheets, commercial ERP systems (including SAP), and bespoke IT systems.
- Performance of the POUT policy has been shown to be within 1% of the optimal *dual base stock* policy (but that policy is non-linear and has no known solution, Boute et al. (2019)).

# The variance ratio and the sum of the squared impulse response

- Both the standard deviation of the inventory and the standard deviation of the orders are required to evaluate the costs.
- The impulse response function directly allows the exact computation of the variance of the system output:

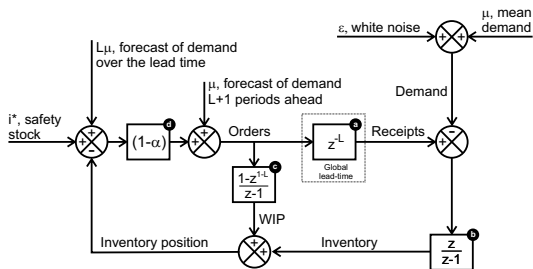
## Lemma (Tsytskin's Relation)

*If the input  $x_t$  to a linear system with impulse response function  $g_t$  is an i.i.d. random process with variance  $\sigma_x^2$ , then the long-run variance of the output  $y_t$  is*

$$\sigma_y^2 = \sigma_x^2 \sum_{t=0}^{\infty} (g_t)^2, \quad (14)$$

*(Tsytskin, 1964).*

# Block diagram and transfer functions of the proportional order-up-to policy



- Rearranging the block diagram for transfer function of the orders,

$$Q[z] = \frac{bd}{1 + abd + cd} = \frac{z - \alpha z}{z - \alpha}. \quad (15)$$

- Similarly, the inventory transfer function is

$$I[z] = (Q[z]a - 1)b = \frac{z(\alpha - z^{1-L}(\alpha + z^L - 1))}{(z-1)(z-\alpha)} \quad (16)$$

# Obtaining the variance of the production orders

- Taking the inverse z-transform of the order transfer function gives the order impulse response function

$$q_t = Z^{-1} \left[ \frac{z - \alpha z}{z - \alpha} \right] = (1 - \alpha)\alpha^t. \quad (17)$$

- Using Tyspkin's Relation the order variance is obtained

$$\sigma_q^2 = \sigma^2 \sum_{t=0}^{\infty} ((1 - \alpha)\alpha^t)^2 = \sigma^2 \left( \frac{1 - \alpha}{1 + \alpha} \right). \quad (18)$$

# Obtaining the variance of the inventory

- Taking the inverse z-transform of the inventory transfer function gives the order impulse response function

$$i_t = Z^{-1} \left[ \frac{z (\alpha - z^{1-L} (\alpha + z^L - 1))}{(z - 1)(z - \alpha)} \right] \quad (19)$$

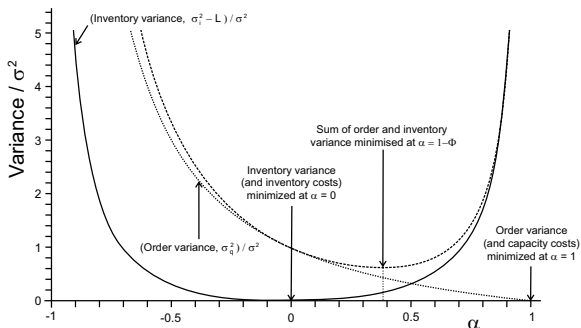
$$= \frac{(\alpha^L - \alpha^{L+t+1} + h[t-1]\alpha^{L+1}(\alpha^t - 1) + h[t-L](\alpha - 1)(\alpha^L - \alpha^{t+1}))}{\alpha^L(\alpha - 1)} \quad (20)$$

Here  $h[t]$  is the Unit Step function. That is,  $h[t < 0] = 0$ , otherwise  $h[t \geq 0] = 1$ .

- Using Tyspkin's Relation the inventory variance is given by

$$\sigma_i^2 = \sigma^2 \left( \sum_{t=0}^{L-1} (-1)^2 + \sum_{t=L}^{\infty} (-\alpha^{t-L+1})^2 \right) = \sigma^2 \left( L + \frac{\alpha^2}{1 - \alpha^2} \right) \quad (21)$$

# The golden feedback controller, $\alpha_\phi$



- Take the derivative of the sum of the variances

$$\frac{d[\sigma_q^2 + \sigma_i^2]}{d\alpha} = -\frac{2((3 - \alpha)\alpha - 1)}{(\alpha^2 - 1)^2} \quad (22)$$

- Use the first order condition to find the golden  $\alpha$

$$\alpha_\phi = \frac{3 - \sqrt{5}}{2} = 2 - \frac{1 + \sqrt{5}}{2} = 1 - \phi \quad (23)$$

# The total cost function

For the OUT policy with associated inventory and production variances  $\sigma_i^2$  and  $\sigma_q^2$ , and a proportional feedback controller  $\alpha$ , the total costs are given by

$$C = C_i^* + C_q^* = \sigma_i(h + b)\phi[z_i] + \sigma_q um\phi[z_q] + u\mu. \quad (24)$$

This can be re-written as

$$C = \varphi(\sigma_i + \lambda(\sigma_q - \sigma_i)) + u\mu, \quad (25)$$

where  $\lambda \in [0, 1]$  indicates the capacity intensity and  $\varphi \geq 0$  the scaling factor,

$$\lambda = um\phi[z_q]/\varphi \text{ and } \varphi = (h + b)\phi[z_i] + um\phi[z_q]. \quad (26)$$

# The cost optimal feedback controller

**Proposition 1 (Optimal  $\alpha^*$ ).** Patterned on Hedenstierna and Disney (2018)

For a given capacity intensity  $\lambda \in [0, 1]$ , the optimal smoothing  $\alpha^* = \lambda$ .

**Proof of Proposition 1 (Optimal  $\alpha^*$ )** The derivative of (25) with respect to  $\alpha$  is

$$\frac{dC}{d\alpha} = \varphi \left( \frac{(1-\lambda) \frac{d\sigma_i^2}{d\alpha}}{2\sigma_i} + \frac{\lambda \frac{d\sigma_q^2}{d\alpha}}{2\sigma_q} \right). \quad (27)$$

The standard derivations have been presented previously, the derivatives are

$$\frac{d\sigma_i^2}{d\alpha} = \frac{2\alpha}{(\alpha^2 - 1)^2}, \quad \text{and} \quad \frac{d\sigma_q^2}{d\alpha} = -\frac{2}{(\alpha + 1)^2}. \quad (28)$$

After substitution and simplification, the first order condition then reveals the stated relation.

# Concluding remarks

- Discrete control theory, z-transforms in particular, can provide a toolkit for understanding the dynamics of supply chain replenishment decisions.
- We can understand stochastic performance via Tyspkin's Relation.
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Thank you for listening

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