Discrete control theory for inventory management: A tutorial

Stephen Disney

International Society for Inventory Research Summer School,
Cardiff Business School, Cardiff, Wales.

July 24-28, 2023

Overview

- Dynamical systems and difference equations
- The z-transform
- The time and the frequency domain
- Test responses: Impulse, step, ramp, parabolic
- Initial value theorem and final value theorem
- The poles and zeros of the transfer function
- Stability
- Aperiodicity
- The Fourier transform and the frequency response
- Random demands and variance analysis
- Bullwhip and lead times
- Order and inventory variance under non-stationary demands

Dynamical systems and difference equations ♠

- Linear discrete-time systems are governed by linear difference equations. For example, consider the following difference equation of auto-regressive demand of the first order,

$$d_t = \phi(d_{t-1} - \mu_d) + \mu_d + \epsilon_t \tag{1}$$

  Here, d_t is the demand at time t, μ_d is the mean demand, $-1 < \phi < 1$ is the auto-regressive parameter, and ϵ_t is an i.i.d. random variable.
- System analysis involves understanding how the system behaves over time for a given input function.
- In the time domain this requires convolution. Discrete convolution,

$$f_t * g_t \stackrel{\text{def}}{=} \sum_{\tau=-\infty}^{\infty} f_\tau g_{t-\tau}, \tag{2}$$

  is a rather complex operation. Here, $*$ is the convolution operator.

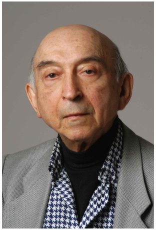
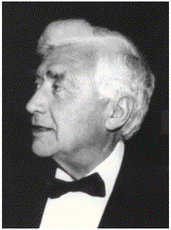
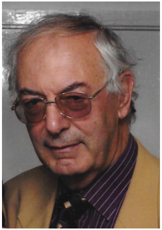
Avoiding convolution in the time domain with z-transforms

- z-transforms allow us to avoid convolution in the time domain by translating the problem into the frequency domain where solutions can be obtained with only addition and multiplication.
- The z-transform is defined by,

$$X[z] = \mathcal{Z}[x_t] = \sum_{t=0}^{\infty} x_t z^{-t}. \tag{3}$$

- z-transforms are the discrete-time analogue of the Laplace transform.

The z-transform family tree

Z-transforms were developed independently during the WWII for military needs:

- In the UK for gun target systems (by **Arnold Tustin**), Bissell (1992a),
- In the US (John Ragazzini and Lotfi Zadeh) for radar,
- In Russia (Yakov Tsypkin), Bissell (1992b).

The z-name originated from the US team, Wikipedia (2018).

Professor **Denis Towill** was a student and colleague of Professor **Arnold Tustin**.

I was a MSc & PhD student, a colleague, and a friend of Professor **Denis Towill**.

Understanding the z-transform: Impulse response

Linearity implies the system output y_t, for any input function x_t, is fully described by the impulse response function g_t. g_t is the solution of the system's difference equation when the input is the unit impulse function $\delta[t]$; $\delta[t=0]=1$ and $\delta[t \neq 0]=0$.

Time, t	$x_t = \delta[t]$	$y_t = g_t$
0	1	1
1	0	0
2	0	0
3	0	0
$\vdots$	0	0

$$G[z] = \sum_{t=0}^{\infty} g_t z^{-t}$$
$$= (1 \times z^0) + (0 \times z^{-1}) + (0 \times z^{-2}) + (0 \times z^{-3}) + \ldots$$
$$= 1$$

Understanding the z-transform: Delay operator

The output of the system can be time shifted one period into the future with the delay operator, z^{-1}.

Time, t	$x_t = \delta[t]$	$y_t = g_t$
0	1	0
1	0	1
2	0	0
3	0	0
$\vdots$	0	0

$$\begin{aligned} G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\ &= (0 \times z^{0}) + (1 \times z^{-1}) + (0 \times z^{-2}) + (0 \times z^{-3}) + \ldots \\ &= z^{-1} \end{aligned}$$

Understanding the z-transform: Scaled responses

We can scale an output with simple multiplication. Here we have combined a scaling operation (multiplying by 0.5) with a delay, z^{-1}.

Time, t	$x_t = \delta[t]$	$y_t = g_t$
0	1	0
1	0	0.5
2	0	0
3	0	0
$\vdots$	0	0

$$\begin{aligned} G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\ &= (0 \times z^0) + (0.5 \times z^{-1}) + (0 \times z^{-2}) + (0 \times z^{-3}) + \ldots \\ &= 0.5 z^{-1} \end{aligned}$$

Understanding the z-transform: Time integration

We can integrate the time domain response with the integration operator $\frac{z}{z-1}$. This is really useful for determining the inventory response.

Time, t	$x_t = \delta[t]$	$y_t = g_t$
0	1	1
1	0	1
2	0	1
3	0	1
$\vdots$	0	1

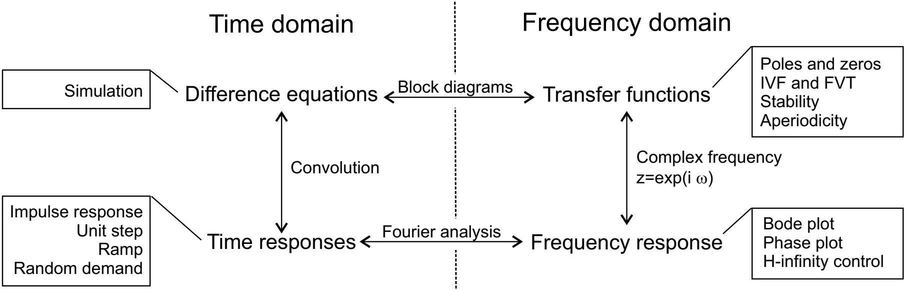
$$
\begin{aligned}
G[z] &= \sum_{t=0}^{\infty} g_t z^{-t} \\
&= (1 \times z^0) + (1 \times z^{-1}) + (1 \times z^{-2}) + (1 \times z^{-3}) + \ldots \\
&= \sum_{i=0}^{\infty} z^{-i} = \frac{z}{z-1}
\end{aligned}
$$

The superposition principle

- All demand processes can be made up in delayed and scaled impulses.
- Due to the **superposition principle**, the output of a linear system to scaled and delayed impulses is the sum of scaled and delayed impulse responses.
- If you understand the unit impulse response then you understand how the (linear) system will react to any demand.
- Consider this example...

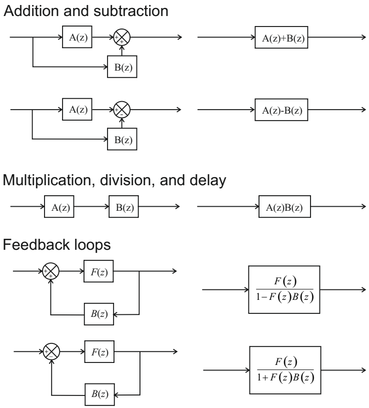
Time t	$\tilde{d}_t$	$\tilde{f}_t$	d_t	f_t	$0.5\tilde{f}_t$	$2\tilde{f}_{t-1}$	$0.5\tilde{f}_t + 2\tilde{f}_{t-1}$	$f_t - (0.5\tilde{f}_t + 2\tilde{f}_{t-1})$
-1	0	0	0	0	0	0	0	0
0	1	0.5	0.5	0.25	0.25	0	0.25	0
1	0	0.25	2	1.125	0.125	1	1.125	0
2	0	0.125	0	0.5625	0.0625	0.5	0.5625	0
3	0	0.0625	0	0.28125	0.03125	0.25	0.28125	0
4	0	0.03125	0	0.140625	0.015625	0.125	0.140625	0
5	0	0.015625	0	0.070313	0.007813	0.0625	0.070313	0

The time and the frequency domain

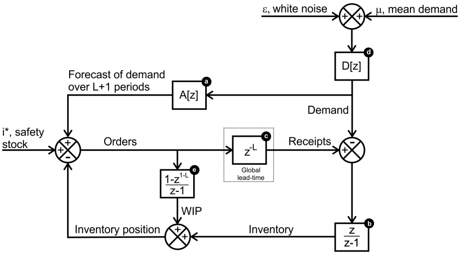


Block diagrams: A visualisation and manipulation tool ♠

Method 1: The rules of block diagram manipulation.



Method 2: Follow the paths. (Sum of all paths from ϵ to state variable of interest)/(1-sum of all paths from state variable, back to itself).



Example: Orders, $\dfrac{O(z)}{\epsilon(z)} = \dfrac{ad+bd}{1+e+bc}$.

Going back to the time domain: Inverse z-transform ♠

Inverse z-transform by direct inversion

The inverse z-transform is given by

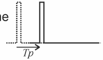
$$f_t = \frac{1}{2\pi i} \oint_c F(z) z^{n-1} dz. \tag{4}$$

This requires the use of Residue Theory and Complex Analysis.

Other approaches:

- Long division
- Partial fraction expansion and matching to standard forms in tables of transform pairs
- Software such as Mathematica and Matlab
- www.wolframalpha.com

Table of z-transform pairs, Disney and Lambrecht (2008)

Demand		Laplace transform	z- transform	Component	Laplace transform	z- transform
Impluse		1	1	Pure time delay 	e^{-Tps}	z^{-Tp}
Step		$\frac{1}{s}$	$\frac{1}{1-z^{-1}}=\frac{z}{z-1}$	n^{th} order lag	$\left(\frac{1}{1+\frac{T_p}{n}s}\right)^n$	$\left(\frac{z}{T_p(z-1)+z}\right)^n$
Ramp		$\frac{1}{s^2}$	$\frac{z^2}{(z-1)^2}$	Integrator	$\frac{1}{s}$	$\frac{1}{1-z^{-1}}=\frac{z}{z-1}$
Parabolic		$\frac{1}{s^3}$	$\frac{z^3}{(z-1)^3}$	WIP integrator (for pure time delays only)	$\frac{1-e^{-T_p s}}{s}$	$\sum_{k=1}^{T_p} z^{-k}=\frac{1-z^{-T_p}}{z-1}$
Sin (at)		$\frac{a}{s^2+a^2}$	$\frac{z^{-1}\sin(a)}{1-2z^{-1}\cos(a)+z^{-2}}$	Exponential smoothing forecast	$\frac{1}{1+T_a s}$	$\frac{z}{z+T_a(z-1)}=\frac{z\alpha}{z+\alpha-1}$
				Moving average forecast of n periods	$\frac{1-e^{-ns}}{ns}$	$\sum_{i=0}^{n-1}\frac{z^{-i}}{n}=\frac{z-z^{1-n}}{n(z-1)}=\frac{1-z^{-n}}{n(1-z^{-1})}$

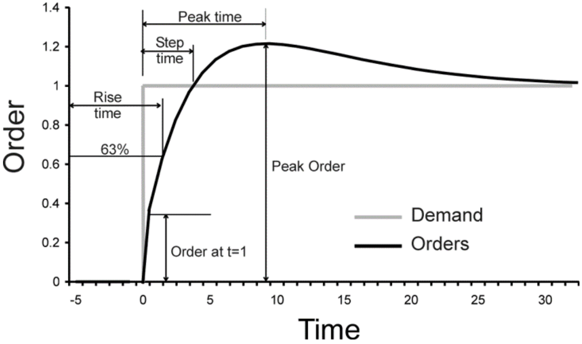
ARIMA demand processes	z- transform
AR(1)	$\frac{z}{z-\phi}$
MA(1)	$\frac{z-\theta}{z}$
ARMA(1,1)	$\frac{z-\theta}{z-\phi}$
ARIMA(p,d,q)	$\frac{1-\sum_{i=1}^{q}\theta_i z^{-i}}{1-\sum_{j=1}^{p}\phi_j z^{-j}-\sum_{k=1}^{d} z^{-k}}$

Properties	Laplace transform	z-transform
Initial value theorem	$\lim_{s\to 0} sF(s)=\lim_{t\to 0^+} f(t)$	$\lim_{z\to\infty} F(z)=f(0)$
Final value theorem	$\lim_{s\to\infty} sF(s)=\lim_{t\to\infty} f(t)$	$\lim_{z\to 1}(z-1)F(z)=f(\infty)$
Convolution	$F_1(s)F_2(s)\xleftrightarrow{L} f_1(t)*f_2(t)$	$F_1(z)F_2(z)\xleftrightarrow{Z} f_1(t)*f_2(t)$
Time product	$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F_1(\zeta)F_2(s-\zeta)d\zeta\xleftrightarrow{L} f_1(t)f_2(t)$	$\frac{1}{2\pi j}\oint_C F_1(\zeta)F_2\left(\frac{z}{\zeta}\right)\zeta^{-1}d\zeta\xleftrightarrow{Z} f_1(t)f_2(t)$

Software tools

Test responses: Impulse, step, ramp, parabolic responses

- The step response is the integral of the impulse response.
- The ramp response is the integral of the step response.
- The parabolic response is the integral of the ramp response.
- These standard test inputs are frequently used by control engineers to qualitatively understand the the nature of the dynamic response of a system.



Initial value theorem (IVT) and final value theorem (FVT)

The IVT and FVT is a useful cross-check of a behaviour of a dynamic system as it provides information on the initial conditions (IVT) and the long run, steady state, behaviour (FVT).

Initial value theorem

The **initial value** is the value of f_t at $t = 0$. It is given by

$$\lim_{z \to \infty} F(z) = f_0 \tag{5}$$

Final value theorem

The **final value** is the value of f_t at $t = \infty$. It is given by

$$\lim_{z \to 1} (z-1)F(z) = f_\infty \tag{6}$$

In situations which seem to be indeterminate, l'Hôpital's rule can be used to take the limit.

l'Hôpital's rule: a useful tool when taking limits ♠

- l'Hôpital's rule allows one to evaluate the limits of indeterminate equations using derivatives.

l'Hôpital's rule

The limit of an indeterminate function is equal to the limit of its derivatives,

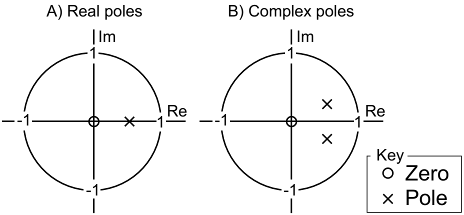
$$\lim_{z \to c} \frac{F(z)}{G(z)} = \lim_{z \to c} \frac{F'(z)}{G'(z)}. \tag{7}$$

- If necessary, l'Hôpital's rule can be applied repetitively until an expression is obtained that can be easily evaluated by substitution.
- Although the rule is often attributed to l'Hôpital, the theorem was first introduced to him in 1694 by the Swiss mathematician Johann Bernoulli.

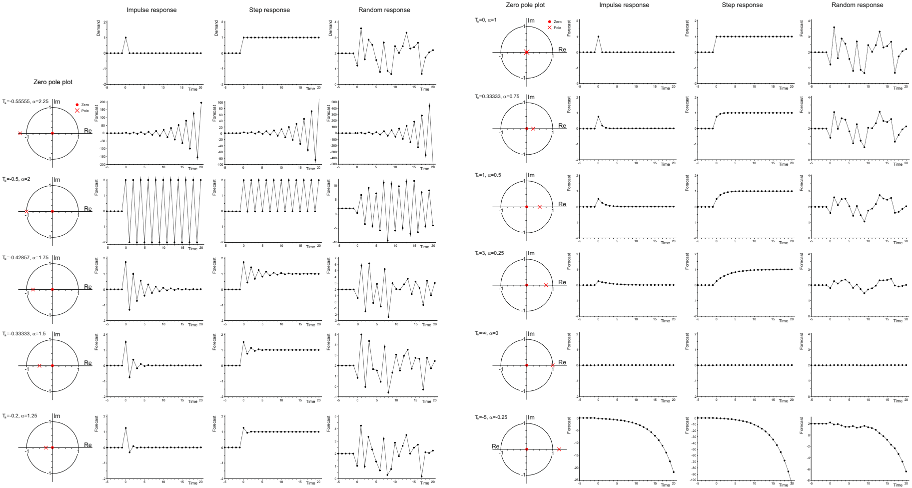
Guillaume de l'Hôpital
1661 – 2 February 1704

Poles and zeros (eigenvalues) of the transfer function ♠

- The roots of the **numerator** of the transfer function are known as the **zeros**.
- The roots of the **denominator** of the transfer function are known as the **poles**.
- Collectively the poles and zeros are known as the **eigenvalues**.
- The poles and zeros can be real or complex.
- The position of the poles and zeros of a system in the complex plane are enough to completely explain the system's dynamic behaviour.



Poles and zeros completely determine the dynamic behaviour: The case of simple exponential smoothing



Stability: definition

- A stable system will react to a finite input and return to steady state conditions in a finite time.
- An unstable system will either diverge exponentially to positive or negative infinity or oscillate with ever increasing amplitude.
- A critically stable system will fall into a limit cycle of constant amplitude to any finite input.
- A system is stable if its poles and zeros lie inside the unit circle in the complex plane, Jury (1974).
- The first order case is simple. Difficulty appears when there are complex poles and/or zeros. These can arise with higher order transfer functions.
- Oscillations in the order rates in supply chain are costly. We must ensure supply chain replenishment rules are stable.

Eliahu Ibrahim Jury
May 23, 1923 – Sept 20, 2020

Jury's stability criterion (1 of 2)

For transfer functions of order n in standard form, $F(z) = \frac{B(z)}{A(z)}$, where

$$B(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0 \quad (8)$$

$$A(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0; \quad a_n > 0. \quad (9)$$

Jury (1974) provides the following necessary and sufficient conditions for stability:

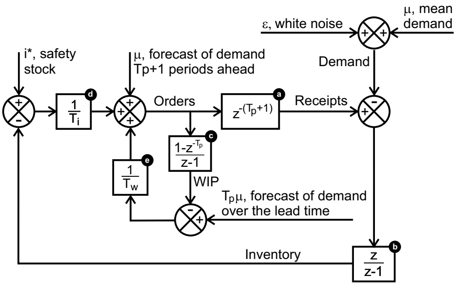
Jury's stability criterion

As system is stable if $A(1) = A(z)|_{z\to 1} > 0$, $(-1)^n A(-1) > 1$ where $(A(-1) = A(z)|_{z\to -1})$, and the $\boldsymbol{\Delta}^{\pm}_{\boldsymbol{n-1}} = \mathbf{X}_{\boldsymbol{n-1}} \pm \mathbf{Y}_{\boldsymbol{n-1}}$ matrices are positive innerwise, where

$$\boldsymbol{X}_{\boldsymbol{n-1}} = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \ldots & a_2 \\ 0 & a_n & a_{n-1} & \ldots & a_3 \\ 0 & 0 & a_n & \ldots & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_n \end{bmatrix}, \quad \boldsymbol{Y}_{\boldsymbol{n-1}} = \begin{bmatrix} 0 & 0 & \ldots & 0 & a_0 \\ 0 & 0 & \ldots & a_0 & \vdots \\ \vdots & 0 & a_0 & \ldots & a_{n-4} \\ 0 & a_0 & a_1 & \ldots & a_{n-3} \\ a_0 & a_1 & a_2 & \ldots & a_{n-2} \end{bmatrix}$$

A matrix is positive innerwise if its determinant is positive and all the determinants of its Inners are also positive.

The Automatic Pipeline, Inventory and Order Based Production Control System has two proportional feedback controllers, John et al. (1994). One feedback controller regulates the discrepancy between target and actual inventory, the other regulates the discrepancy between target and actual WIP, Disney (2008).



The transfer function is given by

$$O(z) = \frac{B(z)}{A(z)} = \frac{T_w z^3}{T_i T_w z^3 + T_i(1 - T_w)z^2 + T_w - T_i}. \quad (10)$$

Stability of APIOBPCS via Jury's criterion (2 of 3)

- $A(1) > 0$ is satisfied if $T_w > 0$.
- $(-1)^n A(-1) > 0$ is satisfied if $T_i > \frac{1}{2}$.
- $\boldsymbol{\Delta}^{\pm}_{\boldsymbol{n-1}}$ is positive innerwise if its determinants and the determinants of its Inners are positive. The $\boldsymbol{\Delta}^{\pm}_{\boldsymbol{n-1}}$ matrices are;

$$\boldsymbol{\Delta}^{+}_{\boldsymbol{n-1}} = \begin{bmatrix} a_3 & a_2 + a_0 \\ a_0 & a_1 + a_3 \end{bmatrix} = \begin{bmatrix} T_i T_w & T_w(1 - T_i) \\ T_w - T_i & T_i T_w \end{bmatrix} \tag{11}$$

$$\boldsymbol{\Delta}^{-}_{\boldsymbol{n-1}} = \begin{bmatrix} a_3 & a_2 - a_0 \\ -a_0 & a_3 - a_1 \end{bmatrix} = \begin{bmatrix} T_i T_w & 2T_i - T_w(1 + T_i) \\ T_i - T_w & T_i T_w \end{bmatrix} \tag{12}$$

We can see that both $\boldsymbol{\Delta}^{\pm}_{\boldsymbol{n-1}}$ are 2x2, thus for our stability analysis here we only need to test whether the determinants of $\boldsymbol{\Delta}^{\pm}_{\boldsymbol{n-1}}$ are positive as there are no inners. The determinants are;

$$|\boldsymbol{\Delta}^{+}_{\boldsymbol{n-1}}| = T_w(T_i(1 + T_i(T_w - 1) + T_w) - T_w) \tag{13}$$

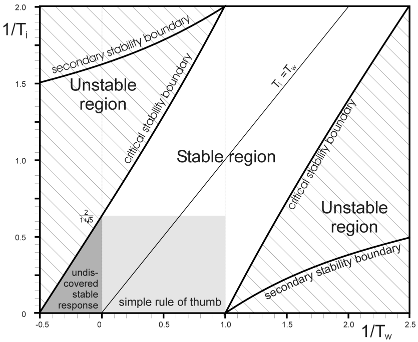
$$|\boldsymbol{\Delta}^{-}_{\boldsymbol{n-1}}| = T_i T_w(3 + T_i) - 2T_i^2 + (T_i(T_i - 1) - 1)T_w^2. \tag{14}$$

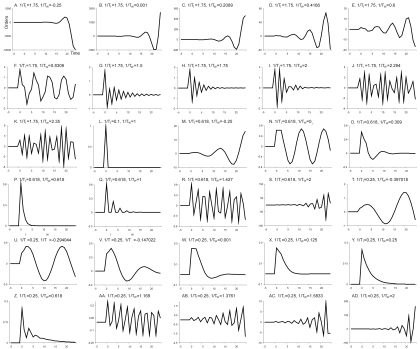
The roots of (13) are completely dominated by the roots of (14). Thus

$$T_i|_c = \frac{T_w^2 - 3T_w \pm T_w\sqrt{1 - 2T_w + 5T_w^2}}{2(T_w - 2 + T_w^2).} \tag{15}$$

completes the critical stability boundary.

Stability of the APIOBPCS (3 of 3), Disney (2008) ♠





- In the left hand panel we have also scaled the axis by the reciprocal of T_i and T_w. Care has to be taken when interpreting the x-axis; as $1/T_w \to 0_+$, $T_w \to +\infty$ and as $1/T_w \to 0_-$, $T_w \to -\infty$. The system is unstable if $-2 < T_w < 0$.
- **Rule of thumb:** the system is always stable if $T_w > 1$ and $T_i > \frac{1+\sqrt{5}}{2} = \Phi$.
- The simulation of the systems impulse response in the right hand panel verifies our stability results. Plot V and W confirms stable solutions exist when $-0.5 < 1/T_w < 0$.

Aperiodicity ♠

- An aperiodic response is a term used to describe a system response with only a finite number (less than n, the order of the system) of maxima or minima in its time domain response.
- Aperiodic systems will not create rogue seasonality and other system induced cycles.
- Aperiodicity occurs when all the roots of the characteristic equation are distinct and lie on the real axis in the interval [0,1) in the z-plane

Aperiodicity in the APIOBPCS model

- When $T_i = T_w$ the transfer function of the production orders becomes

$$O(z) = \frac{z}{T_i z + 1 - Ti} \tag{16}$$

- This system has a single real root at $z = (T_i - 1)/T_i$.
- The APIOBPCS is aperiodic when $T_w = T_i$ and $T_i = [0, \infty)$.

The Fourier transform

- Any time series can be decomposed into a set of **harmonic frequencies**.
- These **harmonic frequencies** are sets of sinusoidal waves with different amplitudes and different phase shifts (translations in time).
- By adding together all of the harmonic frequencies, the original demand signal can be reconstructed.
- The Fourier Transform can be used to identify these harmonic frequencies.
- In our discrete time setting, we can use the discrete time Fourier transform (DTFT), Dejonckheere et al. (2003).

Jean-Baptiste Joseph Fourier
March 21, 1768 - May 5, 1830

The DTFT and the Fast Fourier Transform (FFT)

- The sequence of N numbers, $x_0, x_1, x_2, ..., x_{N-1}$, is transformed into a sequence of N complex numbers, $X_0, X_1, X_2, ..., X_{N-1}$, via the following formula

$$X_k = \sum_{t=0}^{N-1} x_t exp(-i2\pi kt/N) \tag{17}$$

- An efficient algorithm, available in Excel, for computing the DTFT is known as the Fast Fourier Transform (FFT).
- The FFT takes as an input N demands (where N is a power of 2) and produces an output of $N+1$ complex numbers.
- These complex numbers are associated with sinusoidal waves of a certain frequency and the real and imaginary parts of these complex numbers define the amplitude and phase shift of each sinusoid.
- Summing these scaled and time shifted sinusoids together reconstructs the original demand signal.

Reconstructing the signal from the Fourier coefficients

- Define a_k to be the real part of the complex Fourier coefficient X_k; b_k to be the negative of the imaginary part of X_k.
- We may reconstruct the original signal (the demand series) by using the following expressions for each of the harmonic frequencies, h_k

$$h_{k,t} = A \sin\left(F + P\right) \tag{18}$$

with amplitude, $A = \sqrt{a_k^2 + b_k^2}/N$, frequency $F = 2\pi kt/N$ and phase $P = atan2(b_k, a_k)$.

- Summing together all of the harmonic frequencies with

$$x_t = \sum_{k=0}^{N-1} h_{k,t} \tag{19}$$

reproduces the original signal.

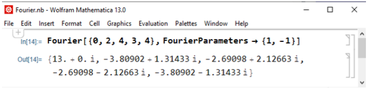
Fourier analysis example (1 of 3)

Consider the following time series of demand

Time, t	Demand, x_t
0	0
1	2
2	4
3	3
4	4

Using (17) we obtain the following five Fourier coefficients, from which we identify the constants a_k and b_k as shown below.

k	Fourier coefficients, X_k	$a_k = Re[X_k]$	$b_k = -Im[X_k]$
0	13	13	0
1	$-3.80902 + 1.31433i$	-3.80902	-1.31433
2	$-2.69098 + 2.12663i$	-2.69098	-2.12663
3	$-2.69098 - 2.12663i$	-2.69098	2.12663
4	$-3.80902 - 1.31433i$	-3.80902	1.31433



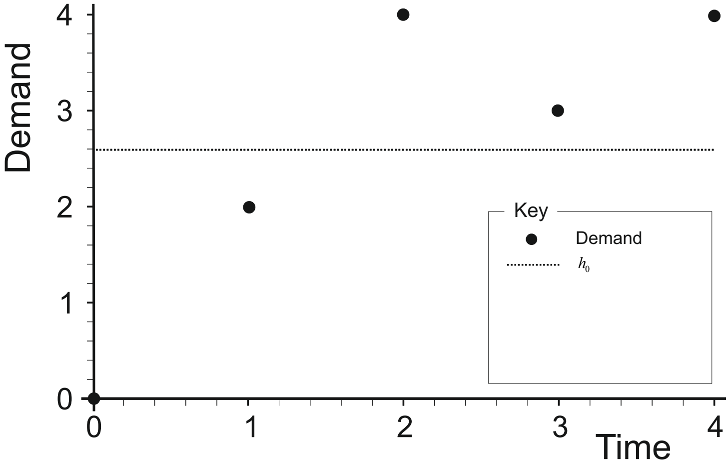
Fourier analysis example (2 of 3)

For each of the harmonic frequencies, we calculate the amplitude and phase shift using (18) to determine the contribution of each harmonic to the demand in each time period.

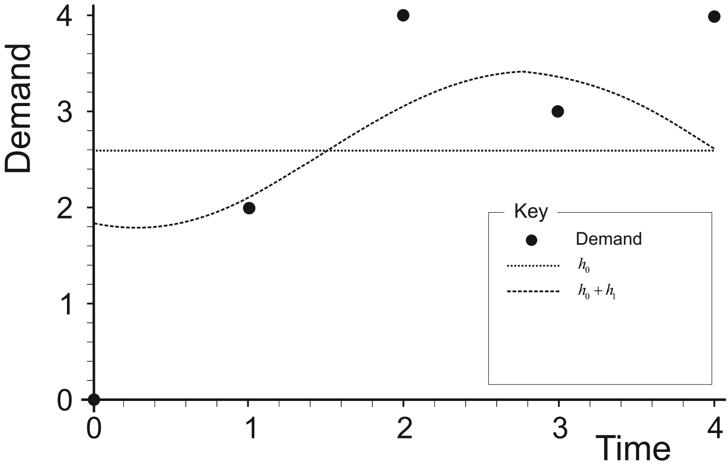
k	Amplitude, $A =$ $\frac{1}{N}\sqrt{a_k^2 + b_k^2}$	Phase, $P =$ $atan2(b_k, a_k)$	Harmonics $h_{k,t} = A \sin(2\pi kt/N + P)$				
			$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
0	2.6	1.5707	2.6	2.6	2.6	2.6	2.6
1	0.8058	-1.903	-0.7618	-0.4854	0.4618	0.7708	0.0145
2	0.6859	-2.239	-0.5381	0.1854	0.2382	-0.5708	0.6854
3	0.6859	-0.9020	-0.5381	0.1854	0.2382	-0.5708	0.6854
$N = 4$	0.8058	-1.2385	-0.7618	-0.4854	0.4618	0.7708	0.0145
		Sum	0	2	4	3	4

Finally, in each time period, we sum the harmonics to obtain the original demand signal.

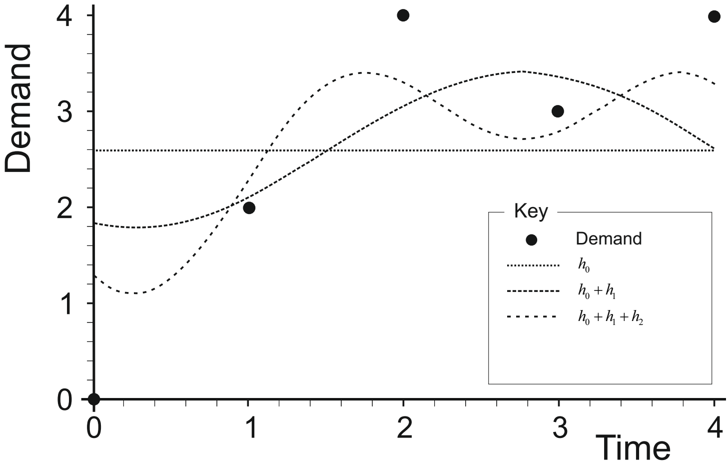
Fourier analysis example (3 of 3)



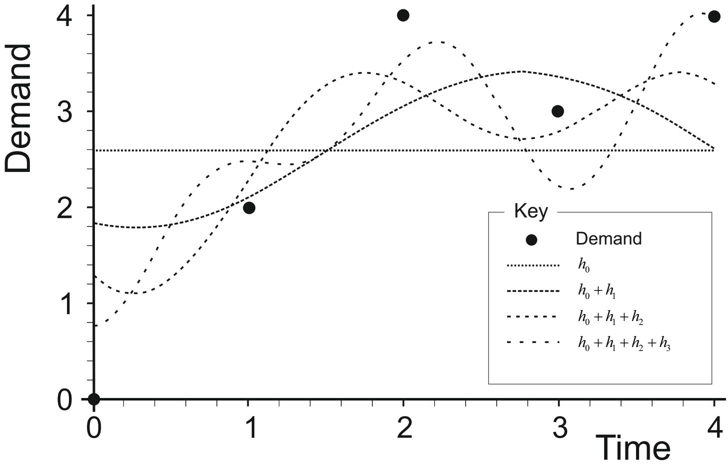
h_0 provides the mean (average) level.



h_1 is a low frequency sin wave which we add to the mean.

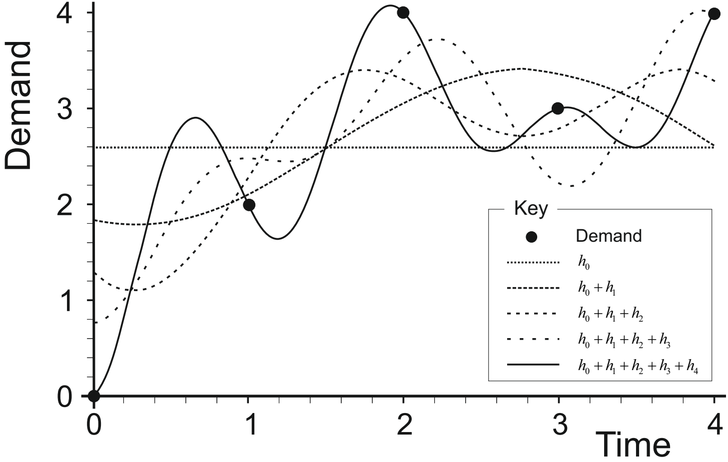


h_2 is a higher frequency sin wave which we add to the sum of h_0 and h_1.



We can ignore the non-integer values of the harmonics.

Fourier analysis example (3 of 3)



The sum of the harmonics reconstructs the demand.

From Fourier transforms to the frequency response

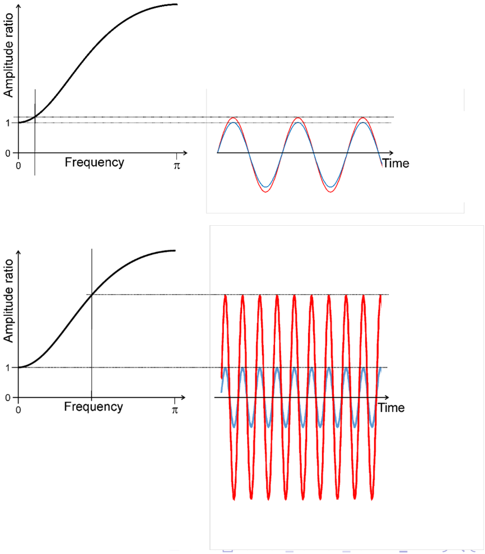
- Any time series can be decomposed into a sum of sinusoidal **harmonic frequencies**.
- In a linear system, if we can understand how a system reacts to each of the individual harmonic frequencies in the demand process, then we can predict how the system will react to the original time series.
- The way to understand how a system reacts to the harmonics is via the **frequency response plot**.
- This can be readily obtained from the z-transform transfer function, $X(z)$, of the system:

$$|AR|^2 = |X(exp(\omega i))|^2 = X(exp(i\omega))X(exp(-i\omega)). \qquad (20)$$

  Here $|AR|$ is the amplitude ratio and ω is the angular frequency measured in radians per time period.
- The amplitude ratio describes how the individual harmonic frequencies are amplified.

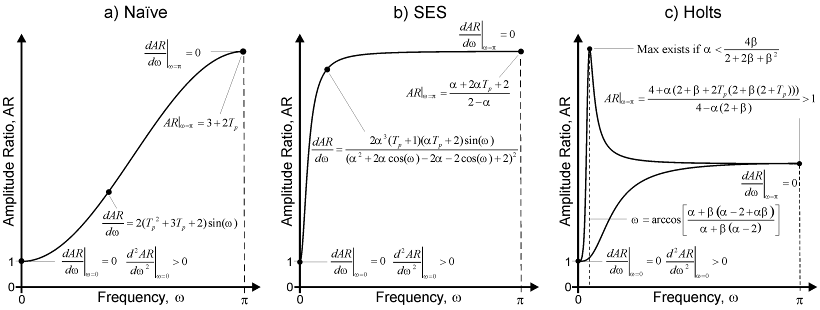
Understanding the frequency response (Bode) plot

- Frequency, is plotted for $\omega \in [0, \pi]$ due to the Nyquist's theorem.
- Blue line is the system input, a sin wave with unit amplitude and frequency ω.
- Red line is the system output, a sin wave with an amplitude ratio AR and frequency ω.
- In this system, the amplitude of every harmonic frequency is amplified.
- *What do you think are the consequences of this?*



Making predictions from the frequency response ♠

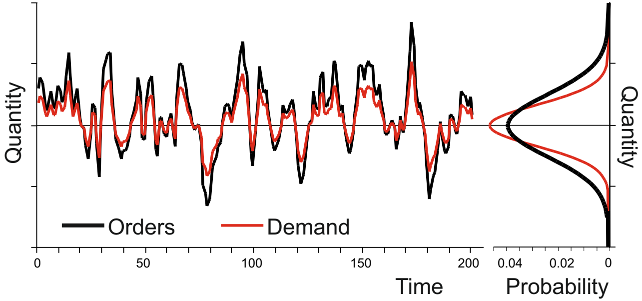
The figures below illustrate the frequency response for the order-up-to (OUT) policy with three different forecasting methods: Naïve, Simple Exponential Smoothing (SES), and Holt's method.



The frequency response plots show that for every possible demand pattern (stationary or non-stationary) the OUT policy will always create bullwhip, for all (constant) lead times, Dejonckheere et al. (2003), Li et al. (2014).

Random demand (a.k.a. linear quadratic control)

- Suppose we have a random demand. In a supply chain, we would often be interested in how the variance of the input (demand) is amplified by the system.



- In the supply chain management field we call this the **bullwhip problem**, or maybe even the **Forrester effect**.
- In control engineering, the problem is known as **linear quadratic control**, or if the noise is Gaussian, **linear Gaussian control**.

Calculating white noise variance ratios

- When the input (demand) into a system is an independently and identically distributed (i.i.d.) random variable, control engineers call this a **white noise** input.
- White noise has an AR of unity for all frequencies. The standard white noise input has a mean of zero and unit variance.
- There are many ways to calculate the (long run) variance of a system's output, given a white noise input:

$$\frac{\mathbb{V}[\text{Output}]}{\mathbb{V}[\text{Input}]} = \frac{1}{\pi}\int_0^{\pi} |F[exp(i\omega)]|^2 d\omega \qquad \text{(Parseval's Relation)}$$

$$= \frac{1}{2\pi i}\oint F[z]F[-z]z^{-1}dz \qquad \text{(Cauchy's Contour Integral)}$$

$$= \sum_{t=0}^{\infty} \tilde{f}_t^2 \qquad \text{(Tsypkin's Relation)}$$

$$= \frac{|\boldsymbol{X_{n+1}} + \boldsymbol{Y_{n+1}}|_b}{a_n|\boldsymbol{X_{n+1}} + \boldsymbol{Y_{n+1}}|} \qquad \text{(Jury's Inners)}$$

Yakov Tsypkin: Discrete Laplace transform

Yakov Zalmanovitch Tsypkin

"He is considered to be the father of pulsed systems in the East. In a series of papers in 1949 and 1950, he extensively developed the discrete Laplace transform (z-transform and modified z-transform) which he applied to the study of pulsed systems. This work culminated in his classic book in this field in 1958." (Bissell, 1992b).

Yakov Zalmanovitch Tsypkin
Sept 19, 1919 - Dec 2, 1997

Tsypkin's sum of the squared impluse response

Tsypkin's relationship. Patterned on Tsypkin (1964, pp183-192) and Boute et al. (2022)

If the input x_t to a linear system with impulse response function $\tilde{g}_t$ is an i.i.d. random process with the variance $\mathbb{V}[x_t]$, then the variance of output is,

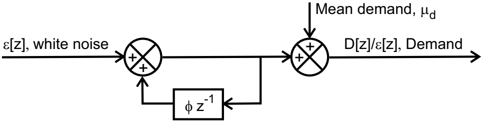
$$\mathbb{V}[y_t] = \mathbb{V}[x_t]\sum_{t=0}^{\infty}(\tilde{g}_t)^2. \tag{21}$$

Proof. Denote $\lim_{t\to\infty}\mathbb{E}[x_t] = \bar{x}$ and $\lim_{t\to\infty}\mathbb{E}[y_t] = \bar{y}$. Taking expectations and limits yields $\bar{y} = \bar{x}\sum_{t=0}^{\infty}\tilde{g}_t$. Indeed, linearity means that a centered input $x_t - \bar{x}$ yields asymptotically centered output $y_t - \bar{y}$. Similarly:

$$\begin{aligned}
\mathbb{V}[y_t] &= \lim_{t\to\infty}\mathbb{E}[(y_t-\bar{y})^2] && \text{(by definition of a variance)}\\
&= \lim_{t\to\infty}\mathbb{E}\left[\left(\sum_{i=0}^{t}(x_i-\bar{x})\tilde{g}_{t-i}\right)^2\right] && \text{(using convolution)}\\
&= \lim_{t\to\infty}\mathbb{E}\left[\left(\sum_{i=0}^{t}(x_i-\bar{x})\tilde{g}_{t-i}\right)\left(\sum_{j=0}^{t}(x_j-\bar{x})\tilde{g}_{t-j}\right)\right] && \text{(expand the square)}\\
&= \lim_{t\to\infty}\sum_{i=0}^{t}\sum_{j=0}^{t}\mathbb{E}[(x_i-\bar{x})(x_j-\bar{x})]\tilde{g}_{t-i}\tilde{g}_{t-j} && \text{(expected value of a sum is the sum of its expected addends)}\\
&= \mathbb{V}[x_t]\sum_{i=0}^{\infty}\tilde{x}_i^2. && (\mathbb{E}[(x_i-\bar{x})(x_j-\bar{x})] = 0 \text{ if } i \neq j \text{ for i.i.d. input } x_t)
\end{aligned}$$

Obtaining the variance ratio: Example of AR(1) demand ♠

The z-transform block diagram of the AR(1) demand process is given by



Manipulating the block diagram yields the system's transfer function,

$$\frac{D[z]}{\epsilon[z]} = \frac{1}{1 - z^{-1}\phi} = \frac{z}{z - \phi}. \tag{22}$$

Taking the inverse z-transform of (22) yields the time domain impulse response,

$$\tilde{d}_t = Z^{-1}\left[\frac{z}{z - \phi}\right] = \phi^t. \tag{23}$$

From Tsypkin's relationship, summing the squared impulse response over all non-negative t, produces an expression for the variance of the AR(1) demand:

$$\mathbb{V}[d_t] = \mathbb{V}[\epsilon_t]\sum_{t=0}^{\infty}(\tilde{d}_t)^2 = \mathbb{V}[\epsilon_t]\sum_{t=0}^{\infty}(\phi^t)^2 = \mathbb{V}[\epsilon_t]\frac{1}{1 - \phi^2}. \tag{24}$$

Bullwhip ratio and $CB[T_p]$, the difference between the order and demand variances, Gaalman et al. (2022)

- Bullwhip exists when the ratio $\sigma_o^2/\sigma_d^2 > 1$. This is equivalent to the difference between the order and demand variance being positive: That is, bullwhip exists if the **critical bullwhip** function $CB[T_p] = \sigma_o^2 - \sigma_d^2 > 1$.
- Using Tsypkin's relation, the demand variance is

$$\sigma_d^2 = \sigma_\epsilon^2 \sum_{t=0}^{\infty} \tilde{d}_t^2 = \sigma_\epsilon^2 \left(\sum_{j=0}^{T_p+1} \tilde{d}_j^2 + \sum_{t=T_p+2}^{\infty} \tilde{d}_t^2 \right) \tag{25}$$

- In the **order-up-to policy, with MMSE forecasting**, the order variance is

$$\sigma_o^2 = \sigma_\epsilon^2 \left(\left(\sum_{j=0}^{T_p+1} \tilde{d}_j \right)^2 + \sum_{t=T_p+2}^{\infty} \tilde{d}_t^2 \right). \tag{26}$$

- Using these variances, $CB[T_p]$ becomes

$$CB[T_p] = \frac{\sigma_o^2 - \sigma_d^2}{\sigma_\epsilon^2} = \left(\sum_{j=0}^{T_p+1} \tilde{d}_j \right)^2 - \sum_{t=0}^{T_p+1} \tilde{d}_t^2. \tag{27}$$

- Bullwhip existence is determined solely by the first $T_p + 1$ demands.

When bullwhip is an increasing function of the lead time

Gaalman et al. (2022) show when the order-up-to policy produces a positive impulse response then the bullwhip effect increases in the lead time.

Theorem 4. Bullwhip lead time behaviour

Iff $\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_{T_p+1}\} > 0$ then $CB[T_p]$ is positive and increasing in the lead time.

Proof $CB[T_p]$ is positive and increasing in T_p if $CB[0] > 0$ and $\forall\, T_p > 0$, $CB[T_p] - CB[T_p - 1] > 0$

- Note always, $\tilde{d}_0 = 1$.
- $CB[0] = \left(\sum_{j=0}^{1} \tilde{d}_j\right)^2 - \sum_{t=0}^{1} \tilde{d}_t^2 = 2\tilde{d}_0\tilde{d}_1$ is positive if additionally $\tilde{d}_1 > 0$
- $CB[1] - CB[0] = 2(\tilde{d}_0 + \tilde{d}_1)\tilde{d}_2$ is positive if additionally $\tilde{d}_2 > 0$
- $CB[2] - CB[1] = 2(\tilde{d}_0 + \tilde{d}_1 + \tilde{d}_2)\tilde{d}_3$ is positive if additionally $\tilde{d}_3 > 0$
- This process can be continued for all T_p. □

Bullwhip is always increasing in the lead-time iff the demand impulse response is positive for all t.

Exponential smoothing provides the MMSE forecast in Integrated Moving Average, IMA(0,1,1), demand

IMA(0,1,1) demand, Box et al. (1994), is defined as

$$d_0 = \mu + \epsilon_0 \quad (28)$$

$$d_t = d_{t-1} - (1-\alpha)\epsilon_{t-1} + \epsilon_t \quad (29)$$

IMA(0,1,1) demand evolves as follows

$$d_0 = \mu + \epsilon_0 \quad (30)$$

$$d_1 = \mu + \epsilon_0 - (1-\alpha)\epsilon_0 + \epsilon_1 = \alpha\epsilon_0 + \epsilon_1 + \mu \quad (31)$$

$$d_2 = \alpha\epsilon_0 + \epsilon_1 + \mu - (1-\alpha)\epsilon_1 + \epsilon_2 = \alpha(\epsilon_0 + \epsilon_1) + \epsilon_2 + \mu \quad (32)$$

By induction, the following recursion occurs

$$d_t = \alpha\left(\sum_{i=0}^{t-1} \epsilon_i\right) + \epsilon_t + \mu \quad (33)$$

Exponential smoothing provides the MMSE forecast in Integrated Moving Average, IMA(0,1,1), demand

IMA(0,1,1) demand, Box et al. (1994), is defined as

$$d_0 = \mu + \epsilon_0 \tag{28}$$
$$d_t = d_{t-1} - (1-\alpha)\epsilon_{t-1} + \epsilon_t \tag{29}$$

IMA(0,1,1) demand evolves as follows

$$d_0 = \mu + \epsilon_0 \tag{30}$$
$$d_1 = \mu + \epsilon_0 - (1-\alpha)\epsilon_0 + \epsilon_1 = \alpha\epsilon_0 + \epsilon_1 + \mu \tag{31}$$
$$d_2 = \alpha\epsilon_0 + \epsilon_1 + \mu - (1-\alpha)\epsilon_1 + \epsilon_2 = \alpha(\epsilon_0 + \epsilon_1) + \epsilon_2 + \mu \tag{32}$$

By induction, the following recursion occurs

$$d_t = \alpha\left(\sum_{i=0}^{t-1}\epsilon_i\right) + \epsilon_t + \mu \tag{33}$$

Exponential smoothing forecasts are given by

$$\hat{d}_0 = \mu \tag{34}$$
$$d_t = d_{t-1} - (1-\alpha)\hat{d}_{t-1} \tag{35}$$

Exponential smoothing forecasts of IMA(0,1,1) demand evolves as follows

$$\hat{d}_0 = \mu \tag{36}$$
$$\hat{d}_1 = \alpha d_1 + (1-\alpha)\hat{d}_0 + \mu = \alpha\epsilon_0 + \mu \tag{37}$$
$$\hat{d}_2 = \alpha d_2 + (1-\alpha)\hat{d}_1 + \mu = \alpha(\epsilon_0 + \epsilon_1) + \mu \tag{38}$$

By induction, the following recursion occurs

$$\hat{d}_t = \alpha\left(\sum_{i=0}^{t-1}\epsilon_i\right) + \mu \tag{39}$$

Exponential smoothing provides the MMSE forecast in Integrated Moving Average, IMA(0,1,1), demand

IMA(0,1,1) demand, Box et al. (1994), is defined as

$$d_0 = \mu + \epsilon_0 \tag{28}$$
$$d_t = d_{t-1} - (1-\alpha)\epsilon_{t-1} + \epsilon_t \tag{29}$$

IMA(0,1,1) demand evolves as follows

$$d_0 = \mu + \epsilon_0 \tag{30}$$
$$d_1 = \mu + \epsilon_0 - (1-\alpha)\epsilon_0 + \epsilon_1 = \alpha\epsilon_0 + \epsilon_1 + \mu \tag{31}$$
$$d_2 = \alpha\epsilon_0 + \epsilon_1 + \mu - (1-\alpha)\epsilon_1 + \epsilon_2 = \alpha(\epsilon_0 + \epsilon_1) + \epsilon_2 + \mu \tag{32}$$

By induction, the following recursion occurs

$$d_t = \alpha\left(\sum_{i=0}^{t-1} \epsilon_i\right) + \epsilon_t + \mu \tag{33}$$

Exponential smoothing forecasts are given by

$$\hat{d}_0 = \mu \tag{34}$$
$$d_t = d_{t-1} - (1-\alpha)\hat{d}_{t-1} \tag{35}$$

Exponential smoothing forecasts of IMA(0,1,1) demand evolves as follows

$$\hat{d}_0 = \mu \tag{36}$$
$$\hat{d}_1 = \alpha d_1 + (1-\alpha)\hat{d}_0 + \mu = \alpha\epsilon_0 + \mu \tag{37}$$
$$\hat{d}_2 = \alpha d_2 + (1-\alpha)\hat{d}_1 + \mu = \alpha(\epsilon_0 + \epsilon_1) + \mu \tag{38}$$

By induction, the following recursion occurs

$$\hat{d}_t = \alpha\left(\sum_{i=0}^{t-1} \epsilon_i\right) + \mu \tag{39}$$

Notice, the forecasts errors are $\forall t$ equal to the noise ϵ_t, revealing that exponential smoothing provides the MMSE forecast of IMA(0,1,1) demand,

$$d_t - \hat{d}_t = \alpha\left(\sum_{i=0}^{t-1} \epsilon_i\right) + \epsilon_t + \mu - \alpha\left(\sum_{i=0}^{t-1} \epsilon_i\right) - \mu = \epsilon_t. \tag{40}$$

Bullwhip lead time behaviour of OUT policy under IMA(0,1,1) demand with MMSE forecasts ♠

- The z-transform of the IMA(0,1,1) demand is given by

$$\frac{D[z]}{\epsilon[z]} = \frac{1-(1-\alpha)z^{-1}}{1-z^{-1}} = \frac{\alpha-1+z}{z-1} \tag{41}$$

- The inverse z-transform of (46) provides the time domain impulse response

$$\tilde{d}_t = \begin{cases} 1 \text{ if } t=0, \\ \alpha \text{ if } t>0. \end{cases} \tag{42}$$

Bullwhip lead time behaviour of OUT policy under IMA(0,1,1) demand with MMSE forecasts ♠

- The z-transform of the IMA(0,1,1) demand is given by

$$\frac{D[z]}{\epsilon[z]} = \frac{1-(1-\alpha)z^{-1}}{1-z^{-1}} = \frac{\alpha-1+z}{z-1} \tag{41}$$

- The inverse z-transform of (46) provides the time domain impulse response

$$\tilde{d}_t = \begin{cases} 1 \text{ if } t = 0, \\ \alpha \text{ if } t > 0. \end{cases} \tag{42}$$

- As $\alpha \in [0,2)$ then $\forall t$, $\tilde{d}_t \geq 0$. As the demand impulse is always positive we conclude, via Theorem 4, under IMA(0,1,1) demand, the OUT policy with MMSE forecasts produces bullwhip that always increases in the lead time.
- Interestingly, we are able to draw this conclusion despite infinite demand and order variances being present.

Inventory variance maintained by the OUT policy under IMA(0,1,1) demand with MMSE forecasts

- The z-transform of the inventory levels is given by

$$\frac{i[z]}{\epsilon[z]} = \frac{\alpha(T_p(z-1)+z)+z-1}{(z-1)^2} - \frac{z^{T_p+1}(\alpha+z-1)}{(z-1)^2} \quad (43)$$

- The inverse z-transform of (46) provides the time domain impulse response

$$\tilde{i}_t = \begin{cases} -1 - t\alpha \text{ if } t \leq T_p, \\ 0 \text{ if } t > T_p. \end{cases} \quad (44)$$

- Via Tyspkin's relationship the inventory variance is given by

$$\frac{\mathbb{V}[i]}{\mathbb{V}[\epsilon]} = \sum_{t=0}^{T_p}(1+t\alpha)^2 = 1 + T_p + \alpha T_p(T_p+1) + \frac{\alpha^2 T_p(T_p+1)(2T_p+1)}{6} \quad (45)$$

- Despite that infinite demand and order variances are present, the variance of the inventory levels is finite (and its derivation is super cool).

Inventory variance maintained by the OUT policy under IMA(0,1,1) demand with MMSE forecasts

- The z-transform of the inventory levels is given by

$$\frac{i[z]}{\epsilon[z]} = \frac{\alpha(T_p(z-1)+z)+z-1}{(z-1)^2} - \frac{z^{T_p+1}(\alpha+z-1)}{(z-1)^2} \tag{46}$$

- The inverse z-transform of (46) provides the time domain impulse response

$$\tilde{i}_t = \begin{cases} -1-t\alpha \text{ if } t \leq T_p, \\ 0 \text{ if } t > T_p. \end{cases} \tag{47}$$

- Via Tyspkin's relationship the inventory variance is given by

$$\frac{\mathbb{V}[i]}{\mathbb{V}[\epsilon]} = \sum_{t=0}^{T_p}(1+t\alpha)^2 = 1 + T_p + \alpha T_p(T_p+1) + \frac{\alpha^2 T_p(T_p+1)(2T_p+1)}{6} \tag{48}$$

- Despite that infinite demand and order variances are present, the variance of the inventory levels is finite (and its derivation is super cool).

Telescoping, triangular numbers, and the sum of the first T_p squared numbers

- Recall

$$\frac{\mathbb{V}[i]}{\mathbb{V}[\epsilon]} = \sum_{t=0}^{T_p}(1+t\alpha)^2 = (1+0\alpha)^2 + (1+1\alpha)^2 + (1+2\alpha)^2 + \ldots + (1+T_p\alpha)^2 \quad (49)$$

- Expand the squares

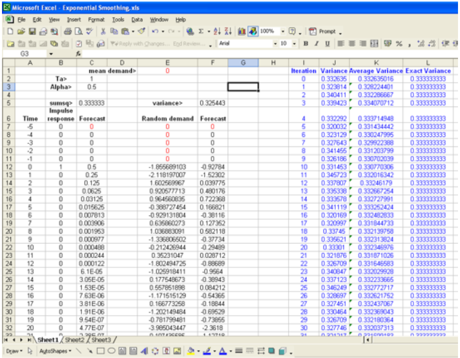
$$\begin{aligned}\frac{\mathbb{V}[i]}{\mathbb{V}[\epsilon]} &= (1+0\alpha)(1+0\alpha) + (1+1\alpha)(1+1\alpha) + \ldots + (1+T_p\alpha)(1+T_p\alpha)\\ &= 1 + 2\alpha(0) + \alpha^2 0^2 + 1 + 2\alpha(1) + \alpha^2 1^2 + 1 + 2\alpha(2) + \alpha^2 2^2 + \ldots\\ &\quad + 1 + 2\alpha(Tp) + \alpha^2 T_p^2. \end{aligned} \quad (50)$$

- Re-order the terms

$$\frac{\mathbb{V}[i]}{\mathbb{V}[\epsilon]} = 1 + Tp + 2\alpha\sum_{i=0}^{Tp} i + \alpha^2 \sum_{i=0}^{T_p} i^2 \quad (51)$$

- There are $T_p + 1$ red terms all equal to unity; the blue terms are the triangular numbers, $\sum_{i=0}^{Tp} i = T_p(T_p+1)/2$. The green terms are the sum of the first T_p squared numbers $\sum_{i=0}^{T_p} i^2 = \frac{\alpha^2 T_p(T_p+1)(2T_p+1)}{6}$. These are both well-known relations.
- Bringing these relations together provides (48).

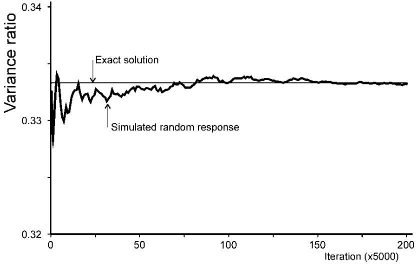
Excel is a very useful tool for building a simulation model where you can verify the internal logic of your system.



Microsoft Excel - Exponential Smoothing.xls

	A	B	C	D	E	F	G	H	I	J	K	L
1			mean demand>		0				Iteration	Variance	Average Variance	Exact Variance
2		Ta>	1						0	0.332635	0.332635016	0.333333333
3		Alpha>	0.5						1	0.323814	0.328224401	0.333333333
4									2	0.340411	0.332286667	0.333333333
5		sumsq>	0.333333		variance>	0.325443			3	0.339423	0.334070712	0.333333333
6	Time	Impulse response	Forecast		Random demand	Forecast			4	0.332292	0.333714948	0.333333333
7	-5	0	0		0	0			5	0.320032	0.331434442	0.333333333
8	-4	0	0		0	0			6	0.323129	0.330247995	0.333333333
9	-3	0	0		0	0			7	0.327643	0.329922388	0.333333333
10	-2	0	0		0	0			8	0.341455	0.331203799	0.333333333
11	-1	0	0		0	0			9	0.326186	0.330702039	0.333333333
12	0	1	0.5		-1.855689103	-0.92784			10	0.331453	0.330770306	0.333333333
13	1	0	0.25		-2.118197007	-1.52302			11	0.345723	0.332016342	0.333333333
14	2	0	0.125		1.602569967	0.039775			12	0.337807	0.33246179	0.333333333
15	3	0	0.0625		0.920577713	0.480176			13	0.335338	0.332667254	0.333333333
16	4	0	0.03125		0.964560835	0.722368			14	0.333578	0.332727991	0.333333333
17	5	0	0.015625		-0.388727454	0.166821			15	0.341119	0.333252424	0.333333333
18	6	0	0.007813		-0.929131804	-0.38116			16	0.320169	0.332482833	0.333333333
19	7	0	0.003906		0.635860273	0.127352			17	0.320997	0.331844733	0.333333333
20	8	0	0.001953		1.036883091	0.582118			18	0.33745	0.332139758	0.333333333
21	9	0	0.000977		-1.336806502	-0.37734			19	0.335621	0.332313824	0.333333333
22	10	0	0.000488		-0.212426944	-0.29489			20	0.33301	0.332346976	0.333333333
23	11	0	0.000244		0.35231047	0.028712			21	0.321876	0.331871026	0.333333333
24	12	0	0.000122		-1.802494725	-0.88689			22	0.326709	0.331646583	0.333333333
25	13	0	6.1E-05		-1.025918411	-0.9564			23	0.340847	0.332029928	0.333333333
26	14	0	3.05E-05		0.177548673	-0.38943			24	0.337123	0.332233665	0.333333333
27	15	0	1.53E-05		0.557851898	0.084212			25	0.346249	0.332772717	0.333333333
28	16	0	7.63E-06		-1.171515129	-0.54365			26	0.328697	0.332621752	0.333333333
29	17	0	3.81E-06		0.166773258	-0.18844			27	0.327451	0.332437067	0.333333333
30	18	0	1.91E-06		-1.202149484	-0.69529			28	0.330464	0.332369043	0.333333333
31	19	0	9.54E-07		-0.781799481	-0.73855			29	0.326709	0.332180364	0.333333333
32	20	0	4.77E-07		-3.985043447	-2.3618			30	0.327746	0.332037313	0.333333333

Sheet1 Sheet2 Sheet3



I always do my analysis using at least two different methods: Excel simulation, maths by hand, Mathematica, R and/or R-shiny. This way I can be sure I am not studying nonsense and wasting my time.

Concluding remarks (from my PhD grandfather)

> THE 'THEORY OF CONTROL SYSTEMS' IN ENGINEERING IS NOW A WELL-DEVELOPED SUBJECT, MAKING USE OF SOME REMARKABLY POWERFUL CONCEPTS AND METHODS OF ANALYSIS, ESPECIALLY IN RELATION TO PROBLEMS OF STABILIZATION AND THE PREVENTION OF UNWANTED OSCILLATIONS.
>
> - ARNOLD TUSTIN -

Thank you for listening

Discrete control theory for inventory management: A tutorial

Stephen Disney

Bibliography

Bissell, C. 1992a. Pioneers of control: An interview with Arnold Tustin. *IEE Review* **June** 223–226.

Bissell, C. 1992b. Yakov Tyspkin: A Russian life in control. *IEE Review* **Sept** 313–316.

Boute, R. N., S. M. Disney, J. Gijsbrechts, J. A. Van Mieghem. 2022. Dual sourcing and smoothing under nonstationary demand time series: Reshoring with SpeedFactories. *Management Science* **68**(2) 1039–1057.

Box, G. E. P., G. M. Jenkins, G. C. Reinsel. 1994. *Time Series Analysis, Forecasting and Control*. 3rd ed. Holden-Day, San Francisco.

Dejonckheere, J., S.M. Disney, M.R. Lambrecht, D.R. Towill. 2003. Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research* **147**(3) 567 – 590.

Disney, S.M. 2008. Supply chain aperiodicity, bullwhip and stability analysis with Jury's inners. *IMA Journal of Management Mathematics* **19**(2) 101–116. doi:10.1093/imaman/dpm033.

Disney, S.M., M.R. Lambrecht. 2008. On replenishment rules, forecasting and the bullwhip effect in supply chains. *Foundations and Trends in Technology, Information and Operations Management* **2**(1) 1–80.

Gaalman, G., S. M. Disney, X. Wang. 2022. When bullwhip increases in the lead time: An eigenvalue analysis of ARMA demand. *International Journal of Production Economics* **250** 108623.

John, S., M.M. Naim, D.R. Towill. 1994. Dynamic analysis of a WIP compensated decision support system. *International Journal of Manufacturing Systems Design* **1**(4) 283–297.

Jury, E.I. 1974. *Inners and the Stability of Dynamic Systems*. Wiley, New York, NY.

Li, Q., S.M. Disney, G. Gaalman. 2014. Avoiding the bullwhip effect using damped trend forecasting and the order-up-to replenishment policy. *International Journal of Production Economics* **149** 3 – 16.

Tsypkin, Y. Z. 1964. *Sampling Systems Theory and its Application, Vol. 2*. Pergamon Press, Oxford.

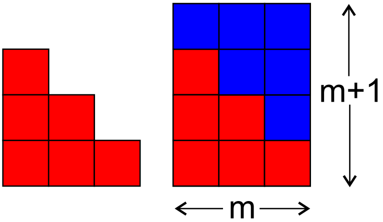
Wikipedia. 2018. $z-$transform. https://en.wikipedia.org/wiki/Z-transform.

The triangular numbers

We wish to show that

$$\sum_{i=0}^{m} i = 1 + 2 + 3 + \ldots + m = \frac{m(m+1)}{2} = S_1. \tag{52}$$

Proof. The following visualisation (when $m = 3$) is convincing.



The sum of the first m square numbers: A proof

We wish to show that

$$\sum_{i=0}^{m} i^2 = \frac{m(m+1)(2m+1)}{6} = S_2. \quad (53)$$

For no apparent reason, consider

$$(n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1 \quad (54)$$

The sum of the first m square numbers: A proof

We wish to show that

$$\sum_{i=0}^{m} i^2 = \frac{m(m+1)(2m+1)}{6} = S_2. \tag{53}$$

For no apparent reason, consider

$$(n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1 \tag{54}$$

For different n we have

$$\begin{cases} n = 1 & 2^3 - 1^3 = 3(1)^2 + 3(1) + 1 \\ n = 2 & 3^3 - 2^3 = 3(2)^2 + 3(2) + 1 \\ n = 3 & 4^3 - 3^3 = 3(3)^2 + 3(3) + 1 \\ \quad\vdots & \qquad\vdots \\ n = m-1 & m^3 - (m-1)^3 = 3(m-1)^2 + 3(m-1) + 1 \\ n = m & (m+1)^3 - m^3 = 3(m)^2 + 3(m) + 1 \end{cases} \tag{55}$$

Summing the equations in (55) leaves $(m+1)^3 - 1^3 = 3S_2 + 3m(m+1)/2 + m$. Simple algebra then leads to the required expression, (53).