Dynamic Analysis of an Integrated VMI-Push Supply Chain

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Material and information flow in a traditional supply chain and an integrated VMI-push supply chain (IVMIPSC)

- In a **traditional supply chain**, customers order from a distributor who holds an inventory to satisfy customer demand. The distributor, in turn, orders from a manufacturer who holds an inventory to satisfy distributor demand.

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- In a **traditional supply chain**, customers order from a distributor who holds an inventory to satisfy customer demand. The distributor, in turn, orders from a manufacturer who holds an inventory to satisfy distributor demand.
- In the **IVMIPSC**, end consumer demand is passed directly the manufacturer.
- The manufacturer determines how much to ship to the distributor to manage the distributor's inventory.
- The manufacturer determines how much to produce in each period; all completed production is immediately sent to the distributor.
- No inventory remains at the manufacturer; the distributor does not place orders.

Our contribution

- We evaluate the dynamic and economic performance of the integrated VMI-push supply chain.
- Dynamic performance is evaluated via the bullwhip ratio (order variance/demand variance) and also the net stock amplification ratio (NSAmp, net stock variance/demand variance).
- Economic performance is evaluated by capacity costs and inventory (holding and backlog) costs.
- The proportional order-up-to (POUT) policy moderates the bullwhip produced in the supply chain.
- The impact of first-order autoregressive, AR(1), demand and the distribution and production lead times is considered.

The vendor managed inventory (VMI) literature

- Arguably, Magee (1958) was the first to describe a form of VMI:
  "Customer specifies a:
  - *minimum inventory*
  - *maximum inventory*

  Supplier assumes responsibility for replenishment, using the flexibility the minimum and maximum inventory levels provide to reduce production and transportation costs."

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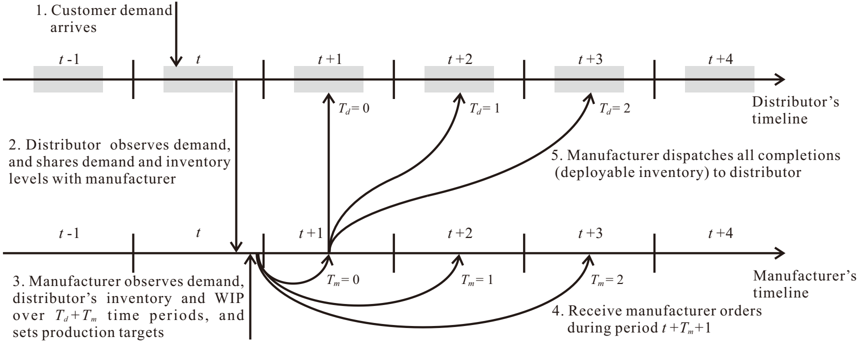
- In the 1980s, Walmart used satellites to send inventory and EPOS data from stores to a central planning team that managed replenishment, Waller et al. (1999).
- Disney (2001) showed that a VMI supply chain acts as a single supply chain echelon.
- Disney and Towill (2003) revealed how a VMI supply chain can be set up to maximise transportation efficiency.
- VMI is now incorporated into ERP systems.
- In push systems, no inventory is kept at the manufacturer; completed production is immediately shipped to the downstream player, Zhao (2019).

Sequence of events in the integrated VMI-push supply chain



- The grey boxes indicate inventory is being held at the distributor; no inventory remains at the manufacturer.
- T_d: The lead time to ship items from the manufacturer to the distributor.
- T_m: The lead time for the manufacturer to produce items.

Customer demand: The AR(1) demand process

We assume the supply chain faces an AR(1) demand process,

$$d_t = \mu_d + \rho(d_{t-1} - \mu_d) + \varepsilon_t, \tag{1}$$

Here,

- μ_d is the mean demand,
- $-1 < \rho < 1$ is the auto-regressive parameter,
- ε_t is a sequence of i.i.d. random variables, with zero mean and finite variance $\mathbb{V}[\epsilon_t]$.

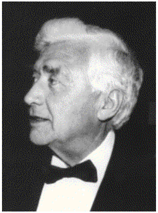
Variance of the AR(1) demand, $\mathbb{V}[d_t]$

Box et al. (2008) show the variance of an AR(1) process is

$$\mathbb{V}[d] = \mathbb{V}[\epsilon_t]\frac{1}{1-\rho^2}. \tag{2}$$

Variances and the sum of the squared impulse response

Yakov Zalmanovitch Tsypkin is considered to be the father of pulsed systems in the East. In a series of papers in 1949 and 1950, he extensively developed the discrete Laplace transform (z-transform and modified z-transform) which he applied to the study of pulsed systems. This work culminated in his classic book in this field in 1958, (Bissell, 1992).



Yakov Tsypkin
Sept 19, 1919 - Dec 2, 1997

Tsypkin's relationship

If the input x_t to a linear system with impulse response function $\tilde{g}_t$ is an i.i.d. random process with the variance $\mathbb{V}[x_t]$, then the variance of output is,

$$\mathbb{V}[y_t] = \mathbb{V}[x_t] \sum_{t=0}^{\infty} (\tilde{g}_t)^2. \tag{3}$$

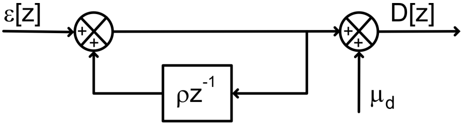
Proof that $\mathbb{V}[y_t] = \mathbb{V}[x_t]\sum_{t=0}^{\infty}(\tilde{g}_t)^2$ patterned on Tsypkin (1964, pp183-192) and Boute et al. (2022)

Proof. Denote $\lim_{t\to\infty}\mathbb{E}[x_t] = \bar{x}$ and $\lim_{t\to\infty}\mathbb{E}[y_t] = \bar{y}$. Taking expectations and limits yields $\bar{y} = \bar{x}\sum_{t=0}^{\infty}\tilde{g}_t$. Indeed, linearity means that a centred input $x_t - \bar{x}$ yields asymptotically centred output $y_t - \bar{y}$. Similarly:

$$\begin{aligned}
\mathbb{V}[y_t] &= \lim_{t\to\infty}\mathbb{E}[(y_t-\bar{y})^2] && \text{(by definition of a variance)}\\
&= \lim_{t\to\infty}\mathbb{E}\left[\left(\sum_{i=0}^{t}(x_i-\bar{x})\tilde{g}_{t-i}\right)^2\right] && \text{(using convolution)}\\
&= \lim_{t\to\infty}\mathbb{E}\left[\left(\sum_{i=0}^{t}(x_i-\bar{x})\tilde{g}_{t-i}\right)\left(\sum_{j=0}^{t}(x_j-\bar{x})\tilde{g}_{t-j}\right)\right] && \text{(expand the square)}\\
&= \lim_{t\to\infty}\sum_{i=0}^{t}\sum_{j=0}^{t}\mathbb{E}[(x_i-\bar{x})(x_j-\bar{x})]\tilde{g}_{t-i}\tilde{g}_{t-j}\\
& && \text{(expected value of a sum is the sum of its expected addends)}\\
&= \mathbb{V}[x_t]\sum_{i=0}^{\infty}\tilde{g}_i^2. && (\mathbb{E}[(x_i-\bar{x})(x_j-\bar{x})] = 0 \text{ if } i \neq j \text{ for i.i.d. input } x_t)
\end{aligned}$$

Obtaining the variance ratio: Example of AR(1) demand

The z-transform block diagram of the AR(1) demand process is given by



Manipulating the block diagram yields the system's transfer function,

$$\frac{D[z]}{\epsilon[z]} = \frac{1}{1 - z^{-1}\rho} = \frac{z}{z-\rho}. \tag{4}$$

Taking the inverse z-transform of (4) yields the time domain impulse response,

$$\tilde{d}_t = Z^{-1}\left[\frac{z}{z-\rho}\right] = \rho^t. \tag{5}$$

From Tsypkin's relationship, summing the squared impulse response over all non-negative t, produces an expression for the variance of the AR(1) demand:

$$\mathbb{V}[d_t] = \mathbb{V}[\epsilon_t]\sum_{t=0}^{\infty}(\tilde{d}_t)^2 = \mathbb{V}[\epsilon_t]\sum_{t=0}^{\infty}(\rho^t)^2 = \mathbb{V}[\epsilon_t]\frac{1}{1-\rho^2}. \tag{6}$$

The manufacturer's forecasts of future AR(1) demand

The minimum mean squared error (MMSE) forecast of the AR(1) end consumer demand, $T_d + 1$ periods ahead, is given by:

$$\hat{d}_{t+T_d+T_m+1|t} = \rho^{T_d+T_m+1}(d_t - \mu) + \mu, \tag{7}$$

The MMSE forecast of demand over the lead time is:

$$\sum_{i=1}^{T_d+T_m} \hat{d}_{t+i|t} = \frac{\rho(\rho^{T_d+T_m} - 1)}{\rho - 1}(d_t - \mu) + \mu(T_d + T_m), \tag{8}$$

The z-transform transfer function of the forecasted demand over is

$$\frac{\hat{D}[z]}{\epsilon[z]} = \rho^{1+T_d+T_m}\frac{D[z]}{\epsilon[z]} = \frac{\rho^{1+T_d+T_m}z}{\rho - z}\frac{D[z]}{\epsilon[z]}. \tag{9}$$

The transfer function of the forecasted demand over the sum of the lead times is

$$\sum_{j=1}^{T_d+T_m} \rho^j \frac{D[z]}{\epsilon[z]} = \frac{\rho(\rho^{T_d+T_m} - 1)}{\rho - 1}\frac{D[z]}{\epsilon[z]}. \tag{10}$$

The manufacturer's replenishment policy: The proportional order-up-to (POUT) policy

The manufacturer dispatches (r_t) all inventory to the distributor. This is the sum of the production completions (the production order initiated $T_m + 1$ periods ago, o_{t-T_m-1}, is completed in period t) plus any finished goods inventory left over from the previous period, f_{t-1}:

$$r_t = o_{t-T_m-1} + f_{t-1}. \tag{11}$$

The manufacturer's inventory f_t is given by

$$f_t = f_{t-1} + o_{t-T_m-1} - r_t = 0. \tag{12}$$

The distributor's inventory i_t is given by

$$i_t = i_{t-1} + r_{t-T_d} - d_t. \tag{13}$$

The manufacturer's production orders, o_t, account for the total inventory and the WIP in the entire supply chain:

$$o_t = \hat{d}_{t+T_d+T_m+1|t} + \beta_m \left(i^\star - i_t + \sum_{j=1}^{T_d+T_m} \hat{d}_{t+j|t} - \sum_{j=1}^{T_d+T_m} o_{t-j} \right). \tag{14}$$

Adding a proportional feedback parameter, β_m to the manufacturers replenishment policy

$$o_t = \hat{d}_{t+T_d+T_m+1|t} + \beta_m \left(i^\star - i_t + \sum_{j=1}^{T_d+T_m} \hat{d}_{t+j|t} - \sum_{j=1}^{T_d+T_m} o_{t-j} \right).$$

- Notice that (14) resembles a proportional order-up-to policy adapted for our integrated VMI-push supply chain, Disney (2008).
- A proportional feedback controller $0 \leq \beta_m < 2$ regulates the inventory and WIP feedback loops, Deziel and Eilon (1967).
- β_m is able to balance, or trade off, the manufacturer's order variance against the distributor's inventory variance.

Economic analysis: Inventory and production costs

- **Inventory costs.** The distributor incurs linear convex inventory holding and backlog costs,

$$C_t^i = h[i_t]^+ + b[-i_t]^+. \tag{15}$$

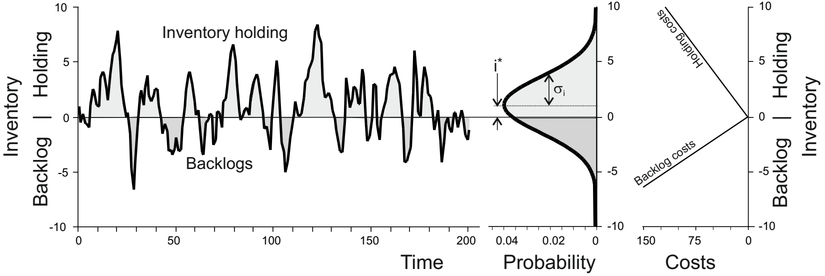
Here h is the per-period, per-unit inventory holding cost, b is the per-period, per-unit backlog costs and $[x]^+ = max[x, 0]$ is the maximum operator.

- **Production costs.** The manufacturer incurs production (capacity) costs of

$$C_t^o = uk + um[o_t - k]^+. \tag{16}$$

Here, u is the unit cost to produce items within normal working hours, whereas um (where $m \geq 1$ is the overtime multipler) is the per-unit cost of producing items in volume-flexible overtime hours (Boute et al., 2022).

Minimising the inventory holding and backlog costs



The expected per-period inventory holding and backlog costs

$$\mathbb{E}[C_t^i] = h\mathbb{E}[[i_t]^+] + b\mathbb{E}[[-i_t]^+], \quad (17)$$

can be minimised by optimising the safety stock $i^\star$ using newsvendor techniques,

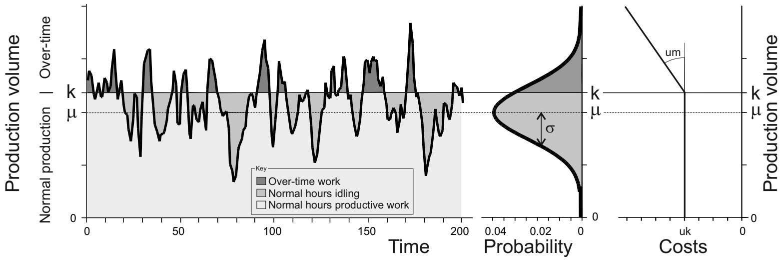
$$i^\star = \operatorname*{argmin}_i \{\mathbb{E}[C_t^i]\} = \sigma_i \Phi^{-1}[h(h+b)^{-1}], \quad (18)$$

resulting in the following minimum inventory and backlog costs,

$$C_i^\star = \min_i \{\mathbb{E}[C_t^i]\} = \sigma_i (h+b)\phi[\Phi^{-1}[h/(h+b)]]. \quad (19)$$

Notes: Here we assume ε_t is normally distributed. $\phi[\cdot]$ and $\Phi[\cdot]$ is the pdf and cdf of the standard normal distribution.

The production costs (aka capacity costs or bullwhip costs)



The expected per-period installed capacity and overtime costs

$$\mathbb{E}[C_t^o] = uk + um\mathbb{E}[[o_t - k]^+], \tag{20}$$

can be minimised by optimising the installed capacity $k^\star$ using newsvendor techniques,

$$k^\star = \underset{k}{\operatorname{argmin}}\{\mathbb{E}[C_t^o]\} = \mu + \sigma_o \Phi^{-1}[m(m-1)^{-1}], \tag{21}$$

resulting in the following minimum production and overtime costs,

$$C_o^\star = \min_k\{\mathbb{E}[C_t^o]\} = u\mu + um\sigma_o\phi[\Phi^{-1}[m/(m-1)]]. \tag{22}$$

Variance of distributor's inventory & manufacturer's orders

Theorem 1: Variances of the inventory ($\mathbb{V}[i]$) and the orders ($\mathbb{V}[o]$).

Using $\kappa_m = 1 - \beta_m$ and $\kappa_\rho = \rho^{T_d + T_m + 1}$, the variance of the distributor's inventory levels is

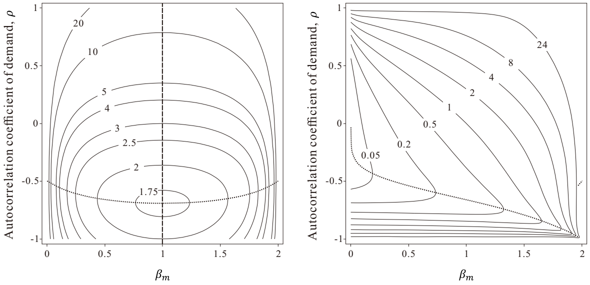
$$\frac{\mathbb{V}[i]}{\mathbb{V}[\varepsilon]} = \frac{1 - \rho^2 + \left((1-\beta_m)^2 - \rho^2\right)\rho^{1+2(T_d+T_m)}}{\beta_m(2-\beta_m)(1-\rho)^3(1+\rho)} + \frac{2\left(\rho - (1-\beta_m)^2\right)\rho^{1+T_d+T_m}}{\beta_m(2-\beta_m)(1-\rho)^3} + \frac{\left(1-\rho^2\right)(T_d + T_m) - \rho(\rho+2)}{(1-\rho)^3(\rho+1)}. \tag{23}$$

The variance of the manufacturer's production orders is

$$\frac{\mathbb{V}[o]}{\mathbb{V}[\varepsilon]} = \frac{\left(\beta_m(\rho+1)(\rho\kappa_m - 1) + 2\beta_m(\rho+1)(\rho-\kappa_m)\kappa_\rho - 2(\rho-\kappa_m)^2\kappa_\rho^2\right)}{(\beta_m-2)(\rho-1)^2(\rho+1)(1-\kappa_m\rho)}. \tag{24}$$

Proof. These variances can be obtained from the difference equations, (1)-(14). **Step 1)** develop a block diagram of the system, Nise (2004). **Step 2)** Manipulate the block diagram for the order and inventory transfer functions. **Step 3)** Take the inverse z-transform to find the time domain impulse response. **Step 4)** The infinite sum of the squared impulse response provides the required variances. □

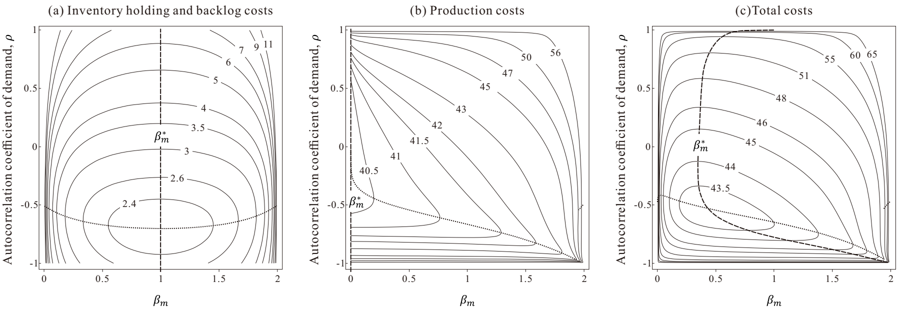
Variance contours when $T_d = T_m = 1$ and $\epsilon = 1$.



(a) Variance of the distributor's inventory

(b) Variance of the manufacturer's orders

- **Panel a**: Variance of the distributor's inventory. $\forall \rho$, $\beta_m = 1$ (dashed line) minimises the inventory costs.
- **Panel b**: Variance of the manufacturer's orders.
- The dotted lines represent the $\{\rho, \beta_m\}$ that minimise each variance.



Note:

- The dashed line means the optimal β_m for different ρ.
- The dotted line indicates the ρ that minimises the costs for a β_m.

♠ Here $h = 1,\ b = 9,\ u = 4,\ m = 1.5,\ \mu = 10,\ \sigma = 1,\ T_d = 1$, and $T_m = 1$.

Economic comparison of the OUT and POUT policy in the integrated VMI-push supply chain.♠

Demand	OUT policy, $\beta_m = 1$			POUT policy, $0 \le \beta_m < 2$			
ρ	$C_i^\star$	$C_o^\star$	$C_i^\star + C_o^\star$	$\beta_m^\star$	$C_i^\star$	$C_o^\star$	$C_i^\star + C_o^\star$
0.9	6.0678	48.1897	54.2575	0.5113	6.6270	46.5177	53.1447
0.5	4.4093	44.0935	48.5029	0.3791	5.0358	42.1572	47.1929
0	3.0397	42.1816	45.2213	0.3535	3.3840	41.0109	44.3949
-0.5	2.3627	41.3726	43.7353	0.4133	2.5479	40.7151	43.2630
-0.9	2.3793	43.3074	45.6867	1.5992	2.6627	42.6682	45.3309

- The OUT policy reduces inventory costs compared to the POUT policy with an optimised feedback controller.
- The production costs are significantly smaller with the POUT policy compared to the OUT policy.
- The total supply chain costs are minimised by the POUT policy.

♠ Here $h = 1$, $b = 9$, $u = 4$, $m = 1.5$, $\mu = 10$, $\sigma = 1$, $T_d = 1$, and $T_m = 1$.

Concluding remarks

- We have studied the dynamic, stochastic, and economic consequences of an integrated VMI-push supply chain.
- We studied the consequences of adopting the POUT replenishment policy for managing the distributor's inventory by regulating the manufacturer's production variance.
- When only inventory costs exist, the OUT replenishment policy minimises costs.
- The POUT policy outperforms the OUT when there are costs associated with the production variance (the bullwhip effect).

Future work

- Future research will compare the performance of our integrated VMI-push supply chain to a traditional supply chain.
- Future research will be concerned with how one could fairly share the benefits of the integrated VMI-push supply chain operations, perhaps with collaborative game theory as explored by Cui et al. (2026).
- Finally, the carbon consequences of adopting the integrated VMI-push supply chains could be considered using the social cost of carbon framework developed by Ma et al. (2025).

Thank you for listening

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(We would like to thank Tingting Wu for inspiring this study)

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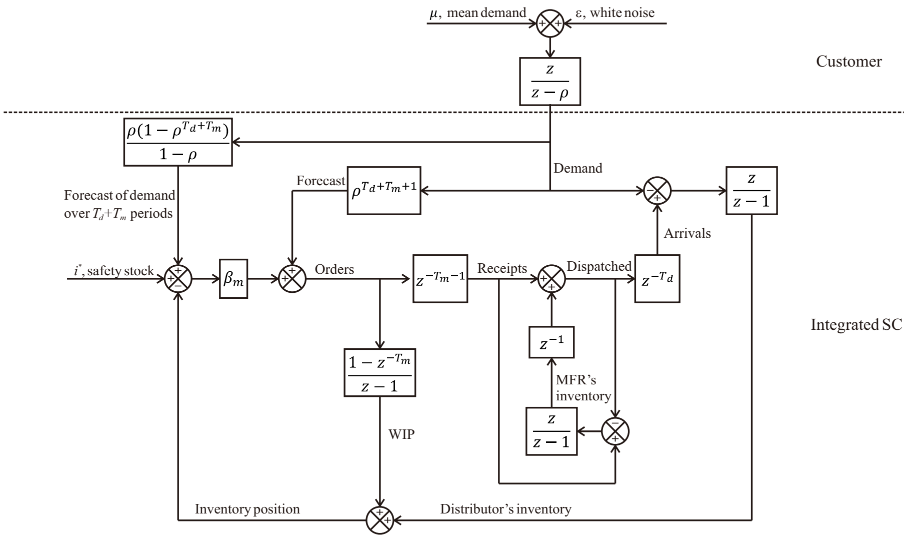
Block diagram of the integrated VMI-push supply chain



Transfer functions for the integrated VMI-push supply chain (1 of 2)

From (13), the inventory balance equation of the distributor, we can obtain the following transfer function of the distributor's inventory, see Nise (2004).

$$\frac{I[z]}{\epsilon[z]} = z^{-1}\frac{I[z]}{\epsilon[z]} + z^{-T_d}\frac{O[z]}{\epsilon[z]} - \frac{D[z]}{\epsilon[z]}. \tag{25}$$

Similarly, the transfer function of the manufacturer's orders is

$$\frac{O[z]}{\epsilon[z]} = \frac{\hat{D}[z]}{\epsilon[z]} + \beta_m\left(\sum_{j=1}^{T_d+T_m}\rho^j\frac{D[z]}{\epsilon[z]} - \sum_{j=1}^{T_d+T_m}z^{-j}\frac{O[z]}{\epsilon[z]} - \frac{I[z]}{\epsilon[z]}\right). \tag{26}$$

The transfer function of the forecasted demand in the period after the lead time was previously given in (9). The transfer function of the forecasted demand over the lead time was given by (10).

Transfer functions for the integrated VMI-push supply chain (1 of 2)

Solving (25) and (26) using (9) and (10) simultaneously provides the transfer functions of the distributor's inventory in the integrated supply chain,

$$\frac{I[z]}{\epsilon[z]} = \frac{(z-1)(\beta_m+\rho-1)\rho^{1+T_d+T_m}+\beta_m(\rho-z)}{z^{T_d+T_m}(\rho-1)(z-1)(\beta_m+z-1)(z-\rho)} - \frac{z^2}{(z-1)(z-\rho)}, \quad (27)$$

and the manufacturer's orders,

$$\frac{O[z]}{\epsilon[z]} = \frac{(z-1)z(\beta_m+\rho-1)\rho^{1+T_d+T_m}+\beta_m z(\rho-z)}{(\rho-1)(\beta_m+z-1)(z-\rho)}. \quad (28)$$

Inventory and order impulse responses for the integrated VMI-push supply chain

By taking the inverse z-transform of (27) and (28), the impulse response functions of the distributor's inventory and the manufacturer's orders are given by

$$\begin{aligned}\tilde{I}_t =& \mathscr{Z}^{-1}\left[\frac{I[z]}{\epsilon[z]}\right] = \frac{\rho^{t+1}-1}{1-\rho} + \\ & \frac{\rho\left((\beta_m+\rho-1)\rho^{T_d+T_m}-\beta_m\right)\left(1-\beta_m-\left(\rho-\beta_m\rho^{t-T_d-T_m}\right)-(1-\rho)(1-\beta_m)^{t-T_d-T_m}\right)H[t-2-T_d-T_m]}{\beta_m(\beta_m+\rho-1)(1-\rho)} + \\ & \frac{\left((\beta_m+\rho-1)\rho^{1+T_d+T_m}-\beta_m\right)\left(1-\beta_m-\left(\rho-\beta_m\rho^{1+t-T_d-T_m}\right)-(1-\rho)(1-\beta_m)^{1+t-T_d-T_m}\right)H[t-1-T_d-T_m]}{\beta_m(\beta_m+\rho-1)(1-\rho)},\end{aligned} \tag{29}$$

and

$$\begin{aligned}\tilde{O}_t =& \mathscr{Z}^{-1}\left[\frac{O[z]}{\epsilon[z]}\right] \\ =& \frac{\left(\beta_m(1-\beta_m)^t\left(\rho^{T_d+T_m+1}-1\right)+(\rho-1)\rho^{t+T_d+T_m+1}\right)H[t-1]+\left((\beta_m+\rho-1)\rho^{T_d+T_m+1}-\beta_m\right)H[-t]}{\rho-1}.\end{aligned} \tag{30}$$

Taking the infinite sum square provides the expressions for the variances of the distributor's inventory and the manufacturer's orders as given by (23) and (24).