THE ROLE OF AN ORDERING POLICY AS AN INVENTORY AND COST CONTROLLER

Takamichi Hosoda*, and Stephen M. Disney
Logistics Systems Dynamics Group, Cardiff Business School, Cardiff University, Aberconway Building, Colum Drive, Cardiff, CF10 3EU, UK
**Corresponding author. Email: hosodat@cardiff.ac.uk , disneysm@cardiff.ac.uk .*

Abstract

We investigate the relationships between an ordering policy, the variance of the inventory levels it maintains and forecasting scheme it exploits. These factors play a major role in the bullwhip effect, an important problem in the field of supply chain management. We start by showing that the order-up-to (OUT) policy is identical to the ordering policy of Vassian (1955). This policy minimizes the variance of the inventory levels subject to the assumptions of a periodic order interval, a constant lead-time and backlogging of excess demand for a particular forecasting policy. These ordering policies are able to match the variance of inventory levels to the variance of the error of the forecasted demand over the lead-time. We compare the performance of the OUT policy with a minimum mean square error (MMSE) forecasting mechanism to that of the moving average (MA) and the exponentially weighted moving average (EWMA) forecasting mechanisms. We also present the myopic inventory policy (e.g. Heyman and Sobel, 1984). This policy minimizes the sum of linear inventory holding and shortage costs. We show that the myopic inventory policy is also equivalent to the OUT policy. Finally, we investigate the relationship between the variance of orders and inventory levels for the three forecasting methods.

Keywords**:** supply chain management, bullwhip effect, inventory variance, cost minimization, order-up-to policy, myopic policy, forecast

Introduction

Arrow *et al*. (1951) introduced the (*S, s*) ordering policy; Karlin (1960) provides the order-up-to (OUT) policy, which is the *s* = *S* case of the (*S, s*) policy. Karlin shows that if the purchase cost is linear, the optimal policy in each period is characterized by a single critical number, *S*, which could vary in successive periods. In 1975, assuming an ARMA (Box *et al*., 1994) demand process, minimum mean square error forecast (MMSE) method, proportional holding and stockout costs and zero lead-time, Johnson and Thompson (1975) shows that the OUT policy is optimal. This policy is often called the myopic inventory policy. Heyman and Sobel (1984) provide an introduction to the myopic policy. On the other hand, Vassian (1955) describes an ordering policy which minimizes the variance of the inventory level for a given forecasting policy. Pinkham (1957) extends Vassian (1955) and presents a linear production rule that achieves an optimal balance between the inventory and the order variance.

Here, we investigate the relationship between the OUT policy, the myopic policy, and Vassian's ordering policy. In the next section we describe the relationship between these policies and some characteristics of them. It has been long recognized that the forecasting method employed has an impact on the performance of an ordering policy (e.g. Badinelli, 1990; Xu *et al*., 2001; Kim and Ryan, 2003; Dejonckheere *et al*., 2003). Thus, we will analyze the effect of three different forecasting methods on the inventory variance ratio. The inventory variance ratio is expressed as $s^2_{\mathrm{inv}}/s^2_{\mathrm{D}}$ (Disney and Towill, 2003a), where s^2_{inv} denotes the variance of inventory level and s^2_{D} denotes the variance of demand. We will also show that the OUT policy is optimal with linear inventory holding and shortage costs and a time delay in replenishment. Furthermore, we will investigate the order variance ratio. This is expressed as $s^2_{\mathrm{O}}/s^2_{\mathrm{D}}$ (Chen *et al*., 2000), where s^2_{O} refers to the variance of orders. This metric is commonly referred to as "bullwhip". We conclude our paper with some managerial insights and provide a short discussion on the trade-off between the inventory and the order variance with different forecasting techniques.

An Ordering Policy to Minimize the Variance of Inventory Level

The OUT policy, which has been used in several research papers (e.g. Lee *et al*., 2000; Chen *et al*., 2000; Alwan *et al*., 2003) can be represented with two equations

$$O_t = D_t + (S_t - S_{t-1}), \tag{1}$$

$$S_t = \hat{D}^l_t + k\hat{s}_l, \tag{2}$$

where; O_t is the order quantity placed at time period t, S_t is the OUT level at time period t, $\hat{D}^l_t$ is an estimate of the total demand over the lead-time (l) made at time period t, and $\hat{s}_l$ is a conditional estimate of the standard deviation of the forecast error over the lead-time. k is a chosen constant to meet a desired service level such as the fill-rate or availability objective.

On the other hand, Vassian's (1955) ordering policy is

$$O_t = \hat{D}^l_t - WIP_t - NS_t + k\hat{s}_l, \tag{3}$$

$$WIP_t = \sum_{i=1}^{l} O_{t-i}, \tag{4}$$

$$NS_t = NS_{t-1} + O_{t-l} - D_t, \tag{5}$$

where; WIP_t denotes the total orders that have already placed but not yet received and NS_t is the net stock level at the end of period t. This ordering policy has received quite a lot of attention recently (e.g. Dejonckheere *et al*., 2003; Disney and Towill, 2003b; Dejonckheere *et al*., 2004). Interestingly, the OUT policy (Eq. 1 and 2), and the ordering policy with WIP (Work In Progress) feedback (Eq. 3 - 5) are equivalent as we will now show. Without loss of generality, we assume $k = 0$, thus $S_t = \hat{D}^l_t$. From Eq. 5, O_t can be written as

$$O_t = NS_{t+l} - NS_{t+l-1} + D_{t+l}. \tag{6}$$

Substituting Eq. 6 into Eq. 4, we obtain the following expression for the WIP,

$$WIP_t = NS_{t+l-1} - NS_t + \sum_{i=1}^{l-1} D_{t+i}. \tag{7}$$

Eq. 3 can be rewritten as

$$O_t = \hat{D}^l_t - WIP_t - NS_t = S_t - \left(NS_{t+l-1} - NS_t + \sum_{i=1}^{l-1} D_{t+i} \right) - NS_t = S_t - \left(NS_{t+l-1} + \sum_{i=1}^{l-1} D_{t+i} \right).$$

Then by using Eq. 3, Eq. 6 and Eq. 7, the second expressions on the right hand side of the above last equation can be rewritten as

$$NS_{t+l-1} = \hat{D}_{t-l} - \sum_{i=1}^{l} D_{t-1+i}. \tag{8}$$

Finally, we will have the final expression of O_t as follows

$$O_t = \hat{D}^l_t - WIP_t - NS_t = S_t - \left(NS_{t+l-1} + \sum_{i=1}^{l-1} D_{t+i} \right) = S_t - \left(\hat{D}^l_{t-1} - \sum_{i=1}^{l} D_{t-1+i} + \sum_{i=1}^{l-1} D_{t+i} \right) = D_t + (S_t - S_{t-1})$$

that we will recognize as being is identical to Eq. 1. Moreover, we can see that the forecast error made at time period $t-l$ is the same as the net stock at time period t. This leads to the conclusion that the variance of the inventory level is equal to the variance of forecast error (Vassian, 1955). From Eq. 8, we find:

$$NS_t = \hat{D}^l_{t-1} - \sum_{i=1}^{l} D_{t-1+i} \ . \tag{9}$$

Eq. 9 confirms an insight that originates in Vassian (1955). The right hand side represents the forecast error over the lead-time. This means that the forecast error made at time period $t-l$ is the same as the net stock inventory at time period t. Therefore, for the infinite time horizon case, the variance of inventory level is equal to the variance of forecast error. Also, this equation shows that the value of the inventory variance depends on the forecasting method exploited in the ordering policy. Vassian shows that an ordering policy with a WIP feedback loop (Eq. 3 - 5) ensures the variance of inventory level is minimal (for the ordering policy employed) and identical to the variance of the error in forecasted demand over the lead-time.

In the next section, we will apply three different types of forecasting methods to a single echelon supply chain model and investigate the inventory variance produced by our OUT policies.

The Forecasting Methodologies and the Inventory Variance Ratio

First, let us describe our single echelon supply chain model. The sequence of events in any period at the echelon is: the order placed earlier is received, and the demand is fulfilled at the beginning of period, the inventory level is reviewed and ordering decision is made at the end of period. In this section, we describe a one echelon supply chain model assuming the OUT policy is used based on the three forecasting methodologies: the moving average (MA), the exponentially weighted moving average (EWMA), and the MMSE forecasting method.

The Demand Model

We assume that the demand pattern follows a first-order autoregressive (AR(1)) process given by

$$D_t = d + r D_{t-1} + e_t, \tag{10}$$

where D_t is the demand at time period t, $-1 < r < 1$ is the autoregressive parameter, and e_t is a i.i.d. normally distributed noise process with mean zero and variance s_e^2. The general expression for the variance of the AR(1) process is

$$s_{\mathrm{D}}^2 = \frac{s_e^2}{1-r^2}.$$

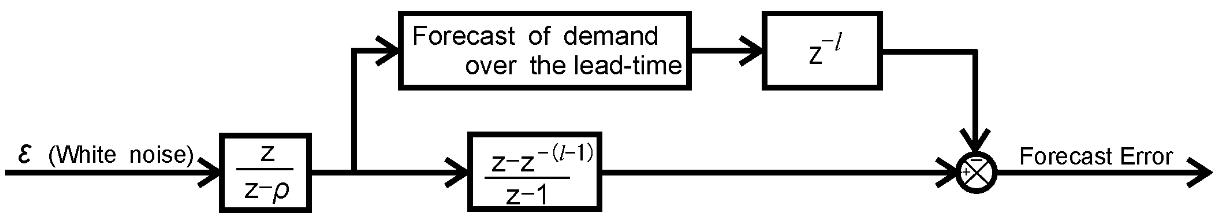

Fig. 1: Block diagram of forecast error with AR(1) demand

Table 1: Transfer functions (Dejonckheere *et al*., 2003, and Hosoda and Disney, 2004)

	MA	EWMA	MMSE
Transfer function of forecast of demand over the lead-time	$\frac{1-z^{-p}}{p(1-z^{-1})} l$	$\frac{a}{1-(1-a)z^{-1}} l$	$\frac{r(r^l-1)}{r-1}$

The Inventory Variance Ratio

Assuming an AR(1) demand process, we may derive the inventory variance ratio using the forecast error variance. From Eq. 9 and 10, a block diagram of the forecast error may be derived as shown in Fig. 1. The transfer functions of the demand over the lead-time, calculated by EWMA, MA and MMSE can be found in Table 1. Using Fig. 1 and Table 1 we may obtain the following expressions for the variance ratio of the inventory levels maintained by the OUT policy with different forecasting techniques. We refer interested readers to Jury (1974) and Disney and Towill (2003a) for more information on how to actually achieve this.

$$\mathrm{VR}_{\mathrm{inv,MA}}=\frac{s^2_{\mathrm{inv,MA}}}{s^2_{\mathrm{D}}}=\frac{l^2\left(p(1-r^2)-2(r-r^{p+1})\right)+lp\left(p(1-r^2)-2(r-r^{l+1})(1-r^p)\right)+2p^2(r^{l+1}-r)}{p^2(r-1)^2},$$

$$\mathrm{VR}_{\mathrm{inv,EWMA}}=\frac{s^2_{\mathrm{inv,EWMA}}}{s^2_{\mathrm{D}}}=\frac{\begin{pmatrix}l^2a(r-1)^2(ar-r-1)+l(a-2)(1-r)\\ \left(1-r^2+ar(2r^l+r-1)\right)+2(a-2)r(ar-r+1)(r^l-1)\end{pmatrix}}{(a-2)(r-1)^2(ar-r+1)},$$

$$\mathrm{VR}_{\mathrm{inv,MMSE}}=\frac{s^2_{\mathrm{inv,MMSE}}}{s^2_{\mathrm{D}}}=\frac{l(1-r^2)+r(1-r^l)(r^{l+1}-r-2)}{(1-r)^2},$$

where; $\hat{s}_{\mathrm{inv}}$ is the variance of inventory level, p is the number of historical data periods used in the MA forecasting scheme, and $0<a\leq 2$ is the exponential smoothing constant. Note that for its own particular forecasting policy the inventory variance is minimized by this policy.

In numerical examples in this paper we will use two lead-time settings: $l = 2$ and $l = 5$. Furthermore setting $p=5$ and $a=2/(p+1)=0.33$, to yield the same average age of data in the EWMA and MA schemes (e.g. Brown, 1963, 106-108) we may plot the inventory variance ratios as in Fig. 2. The MMSE has the lowest variance ratio at any value of r as the variance of forecast error of the MMSE is minimal. The advantage of the MMSE becomes greater when the lead-time becomes longer. When $r>0$, we find that the EWMA shows slightly better performance than that of the MA in terms of the inventory variance ratio.

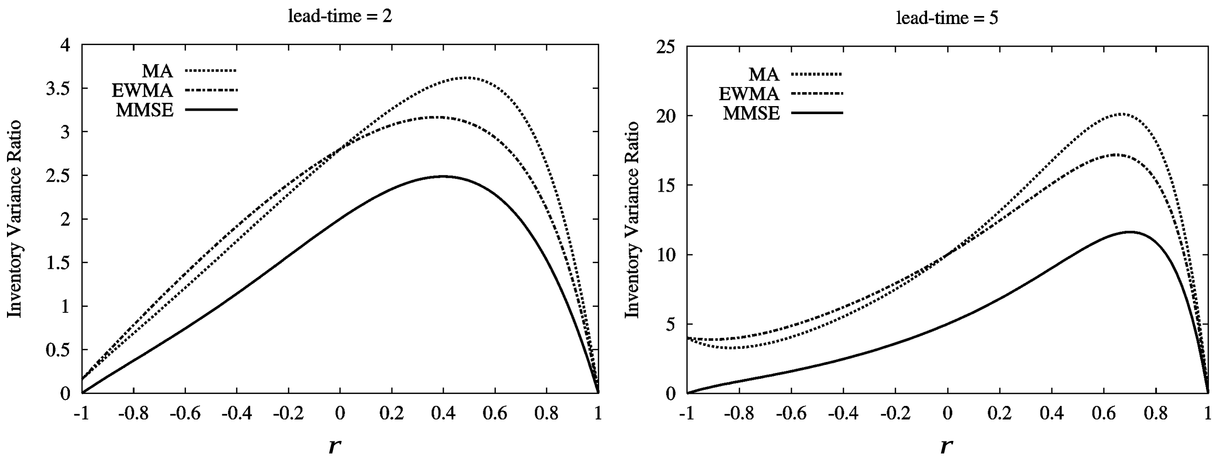

Fig. 2: Inventory variance ratio ($p=5, a=0.33$)

The Myopic Ordering Policy

Here we will show that the structure of our ordering policies are optimal in that they minimize the sum of linear inventory holding and shortage (backlog) costs at every time period for the given forecasting scheme. Let us introduce the myopic inventory policy. The myopic inventory policy is obtained by solving the following equation. Note that we use notation $(x)^{+}$ to describe $max(0, x)$.

$$\min_{S_t} \sum_{t=1}^{\infty} \beta^{t-1} E\left[c(1-\beta)S_t + \beta^{l} g\left(S_t, \hat{D}_t^{l}\right)\right], \tag{11}$$

where

$$g\left(S_t, \hat{D}_t^{l}\right) = h\left(S_t - \hat{D}_t^{l}\right)^{+} + p\left(\hat{D}_t^{l} - S_t\right)^{+},$$

where β is a discount factor, c is a wholesale unit cost, S_t is the OUT level at time period t, h is a holding cost, and π is the penalty cost due to excess demand. By solving this problem, we may find a myopic inventory policy with the optimal order-up-to level, which minimizes Eq. 11 (e.g. Heyman and Sobel, 1984, 75-78):

$$S_t^{*} = Q_{l|\tau_t}^{-1}\left[\frac{\pi - c(1-\beta)/\beta^{l}}{h+\pi}\right],$$

where $Q_{l|\tau_t}$ is the conditional distribution function for $\hat{D}_t^{l}$ given by $\tau_t = \{D_t, D_{t-1}, D_{t-2}, \ldots\}$, the set of all observed demands, with the mean $\hat{D}_t^{l}$ and the standard deviation σ_l. We will recognize that S_t^{*}, the optimal order-up-to level at time period t, can be expressed as,

$$S_t^{*} = \hat{D}_t^{l} + z\sigma_l$$

where; $z = \Phi^{-1}\left(\pi - c(1-\beta)/\beta^{l}\right)/(h+\pi)$ and Φ is the standard normal distribution. If we assume the order quantity that should be placed at time period t is that the total demand at time period time t plus the difference of S_t^{*} and S_{t-1}^{*}, we have the optimal order quantity at time period t as follows:

$$O_t^{*} = D_t + \left(S_t^{*} - S_{t-1}^{*}\right) = D_t + \left(\hat{D}_t^{l} - \hat{D}_{t-1}^{l}\right).$$

The same equation can be found by substituting Eq. 1 into Eq. 2. Therefore, considering that the OUT policy (Eq. 1 and 2), and the ordering policy with WIP feedback (Eq. 3 - 5) are equivalent, then both ordering policies discussed in the paper are myopic optimal policies.

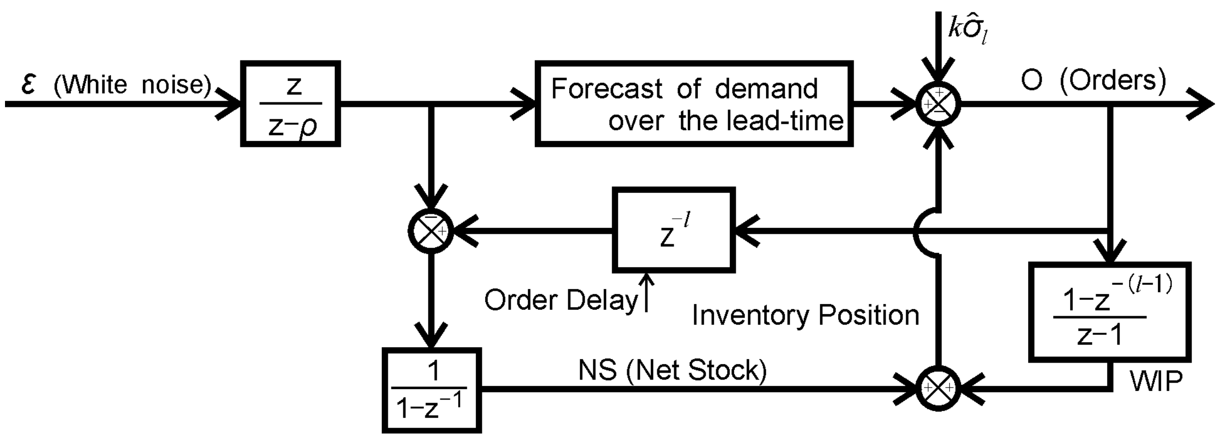

Fig. 3: Block diagram of ordering policy

Order Variance Ratio

It is reasonable to question how the order variance is amplified by our ordering policies as the variance of order rate and inventory levels are inherently liked. We may derive the order variance ratio from Fig. 3 and Table 1. The resulting variance ratios are;

$$\mathrm{VR}_{\mathrm{O,MA}} = \frac{s^2_{\mathrm{O,MA}}}{s^2_{\mathrm{D}}} = 1 + 2(1 - r^p)\left(\frac{l}{p} + \frac{l^2}{p^2}\right),$$

$$\mathrm{VR}_{\mathrm{O,EWMA}} = \frac{s^2_{\mathrm{O,EWMA}}}{s^2_{\mathrm{D}}} = 1 + la\left(2 + \frac{2la}{2 - a}\right)\left(\frac{1 - r}{1 - (1 - a)r}\right),$$

$$\mathrm{VR}_{\mathrm{O,MMSE}} = \frac{s^2_{\mathrm{O,MMSE}}}{s^2_{\mathrm{D}}} = 1 + \frac{2r(1 - r^l)(1 - r^{l+1})}{1 - r}.$$

We have plotted the bullwhip variance ratios in Fig. 4.

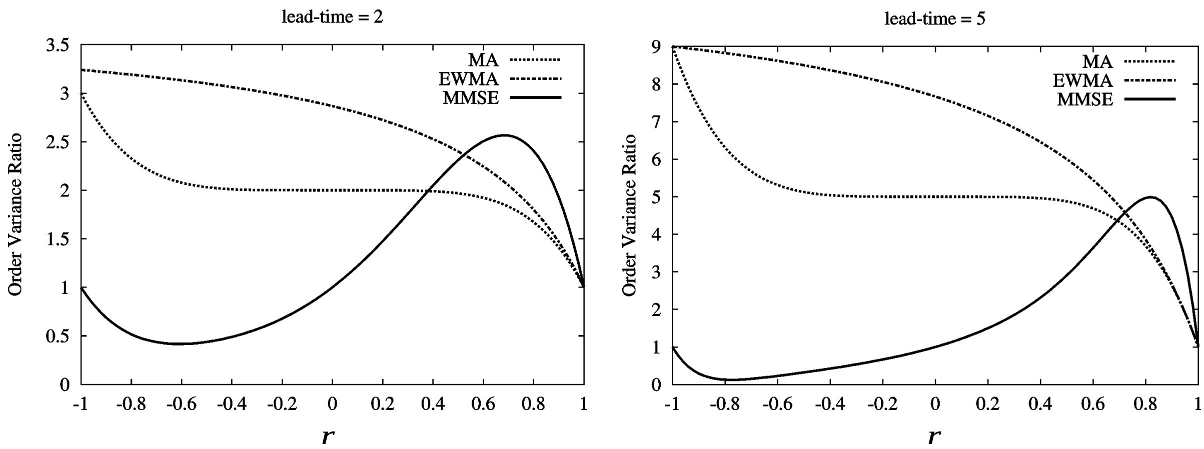

Fig. 4: Order variance ratio ($p = 5$, $a = 0.33$)

Fig. 4 reveals that

- Not always, but for almost all values of r, the MMSE produces the least amount of bullwhip.
- However, in the case of higher value of r, the MMSE may not be the best choice. Instead the simplest method, the MA yields the lowest amount of bullwhip
- The EWMA does not show the lowest amount of bullwhip at our settings of r and l, although for high values of r, it does outperform the MMSE forecasting scheme.

A Trade-off Analysis

Disney *et al*. (2004) investigate the trade-off between the variance of order and inventory levels. They suggest using the following objective function, the weighted sum of a convex combination of a single weight, w,

$$OF = w\mathrm{VR}_{\mathrm{O}} + (1 - w)\mathrm{VR}_{\mathrm{inv}}. \tag{12}$$

We may then search the r - w plane for the optimal forecasting policy to minimize the cost of this objective function. Numerical investigations reveal the following figure (Fig. 5). Here we can see that the MMSE scheme produces superior performance most of the time. However, for high values of auto-regression, MA and EWMA schemes result in superior performance. The EWMA forecasting scheme is surprisingly more popular than our previous figures and insights suggest. The effect of the lead-time seems to increase the effectiveness of the MMSE forecasting scheme. Although the EWMA and the MA schemes are not optimal at minimising the error in the forecast of demand over the lead-time (and hence the inventory variance), they do better than the MMSE scheme when both order and inventory variance is considered in the cost function. Thus, in cases when production variability is expensive then they may, neither the less, be the best forecasting policy to use.

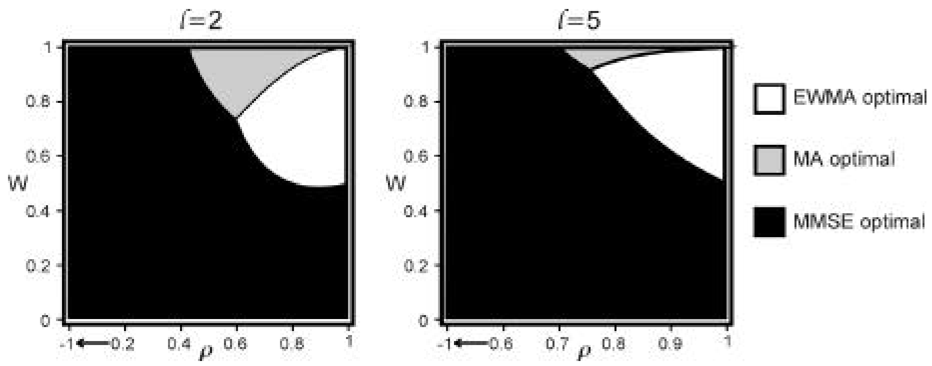

Fig. 5: The optimal forecasting policy in the ρ - w plane, ($p = 5$, $a = 0.33$)

Conclusion

We have presented two ordering policies and highlighted their role as an inventory and cost controller. We have found that the considered ordering policies can minimize not only the variance of inventory level, but also that the sum of linear inventory holding and shortage (backlog) costs. We have also shown that the variance of forecast error is identical to the variance of inventory level. Therefore, a different forecasting method causes different amounts of inventory variance. The MMSE forecast results in the lowest inventory variance ratio at any values of r. However, we find that the MA may exceed the MMSE in terms of bullwhip or the order variance ratio.

We have also investigated the trade-off between the inventory and order variances when different forecasting polices are exploited with the OUT policy. From Fig. 2 and Fig. 4, we extract the variance ratios of the OUT with MMSE forecasting and create Fig. 6. Here VR_{inv} is already relatively large compared with VR_{order}, especially in the longer lead-time case. In general, the difference between VR_{inv} and VR_{order} becomes greater as the lead-time increases (Hosoda and Disney, 2004). Thus, if inventory costs are more important in a certain scenario (e.g. $w < 0.5$), the OUT ordering policy with the MMSE scheme is the ordering policy to use. Fig. 5 also supports this insight. At the same time, interestingly, our results (Fig. 5) show that the MMSE scheme is not optimal for all demand patterns or all weightings between inventory and capacity (order variance) related costs. The concept of this trade-off becomes even more important if we consider a multi-echelon supply chain because the higher order variance may affect other participant's performance in terms of fill-rate, stock holding cost and production cost. Exactly how to manage the trade-off to achieve the maximum performance and minimum cost supply chains is an important avenue for future research.

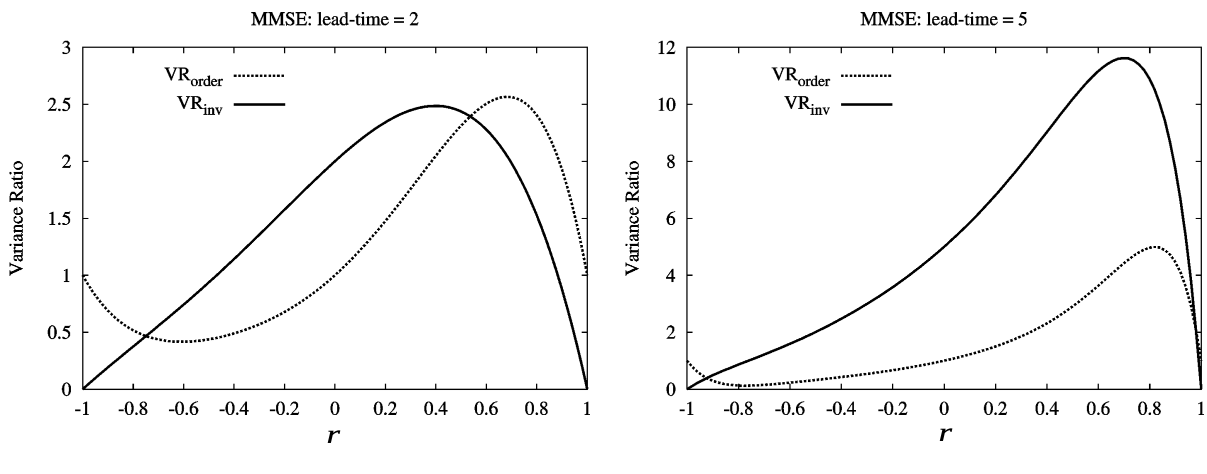

Fig. 6: Variance ratios (MMSE scheme)

From this paper, practical tips for the inventory manager can be obtained:

- OUT policies with the MMSE forecasting minimize the variance of inventory levels over time and result in minimum inventory cost solutions.
- However if order variance costs exist and are dominant in the objective function, and the value of r is high, a different forecasting policy may be able to achieve better performance. This is especially true in the case of a short lead-time.
- Shortening the lead-time will reduce the variance of the inventory levels and the bullwhip produced by the system simultaneously.

References

- Alwan, L., Liu, J., Yao, D., 2003. Stochastic characterization of upstream demand process in a supply chain. *IIR Transactions* 35, 207–219.
- Arrow, K. J., Harris, T., Marschak, J., 1951. Optimal inventory policy. *Econometrica* 19 (3), 250-272.
- Badinelli, R. D., 1990. The inventory costs of common mis-specification of demand-forecasting models. *International Journal of Production Research* 28 (12), 2321-2340.
- Box, G. E., Jenkins, G. M., Reinsel, G. C., 1994. *Time Series Analysis: Forecasting and Control*, 3rd ed. Prentice-Hall, NJ.
- Brown, R. G., 1963. *Smoothing, Forecasting and Prediction of Discrete Time Series*. Prentice-Hall, NJ.
- Chen, Y. F., Drezner, Z., Ryan, J. K., Simchi-Levi, D., 2000. Quantifying the bullwhip effect in a simple supply chain: The impact of forecasting, lead times, and information. *Management Science* 46 (3), 436–443.
- Dejonckheere, J., Disney, S. M., Lambrecht, M. R., Towill, D. R., 2003. Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research* 147, 567–590.
- Dejonckheere, J., Disney, S. M., Lambrecht, M. R., Towill, D. R., 2004. The impact of information enrichment on the bullwhip effect in supply chains: A control engineering perspective. *European Journal of Operational Research* 153, 727–750.
- Disney, S. M., Towill, D. R., 2003a. On the bullwhip and inventory variance produced by an ordering policy. *OMEGA International Journal of Management Science* 31, 157-167.
- Disney, S. M., Towill, D. R., 2003b. The effect of vendor managed inventory (VMI) dynamics on the bullwhip effect in supply chains. *International Journal of Production Economics* 85, 199–215.
- Disney, S. M., Towill, D. R., van de Velde, W., 2004. Variance amplification and the golden ratio in production and inventory control. Forthcoming in the *International Journal of Production Economics*.
- Heyman, D. P., Sobel, M. J., 1984. *Stochastic Models in Operations Research*. Vol. II. McGraw-Hill, NY.
- Hosoda, T., Disney, S. M., 2004. Three echelon supply chain with minimum mean square error forecasting. Working paper, Cardiff Business School, Cardiff University.
- Johnson, G. D., Thompson, H. E., 1975. Optimality of myopic inventory policies for certain dependent demand processes. *Management Science* 21 (11), 1303–1307.
- Jury, E. I., 1974. *Inners and Stability of Dynamic Systems*. John Wiley & Sons, London.
- Karlin, S., 1960. Dynamic inventory policy with varying stochastic demands. *Management Science* 6 (3), 231–258.
- Kim, H., Ryan, J. K., 2003. The cost impact of using simple forecasting techniques in a supply chain. *Naval Research Logistics* 50, 388-411.
- Lee, H. L., So, K. C., Tang, C. S., 2000. The value of information sharing in a two-level supply chain. *Management Science* 46 (5), 626–643.
- Pinkham, R., 1957. An approach to linear inventory-production rules. *Operations Research* 6 (2), 185-189.
- Vassian, H. J., 1955. Application of discrete variable servo theory to inventory control. *Journal of the Operations Research Society of America* 3 (3), 272–282.
- Xu, K., Dong, Y., Evers, P. T., 2001. Towards better coordination of the supply chain. *Transportation Research Part E* 37, 35-54.